

THE TOPOLOGICAL ATIYAH–SEGAL MAP

DANIEL A. RAMRAS

ABSTRACT. Associated to each finite dimensional linear representation of a group G , there is a vector bundle over the classifying space BG . We introduce a framework for studying this construction in the context of infinite discrete groups, taking into account the topology of representation spaces. This involves studying the homotopy group completion of the topological monoid formed by all unitary (or general linear) representations of G , under the monoid operation given by block sum. In order to work effectively with this object, we prove a general result showing that for certain homotopy commutative topological monoids M , the homotopy groups of ΩBM can be described explicitly in terms of unbased homotopy classes of maps from spheres into M .

Several applications are developed. We relate our constructions to the Novikov conjecture; we show that the space of flat unitary connections over the 3-dimensional Heisenberg manifold has extremely large homotopy groups; and for groups that satisfy Kazhdan’s property (T) and admit a finite classifying space, we show that the reduced K -theory class associated to a spherical family of finite dimensional unitary representations is always torsion.

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1. INTRODUCTION

In the 1950s and 1960s, Atiyah and Segal studied a construction that associates a vector bundle to each (complex) representation of a group G . This construction yields a map

$$R[G] \longrightarrow K^*(BG),$$

from the complex representation ring of G to the complex K -theory of its classifying space BG . The simplest form of the Atiyah–Segal Completion Theorem [2, 5] states that when G is a compact Lie group, this map becomes an isomorphism after completing $R[G]$ at its augmentation ideal.

We introduce an analogous map for infinite discrete groups, where the topology of the representation spaces $\mathrm{Hom}(G, \mathrm{GL}_n(\mathbb{C}))$ plays a key role. Loosely speaking, this map assigns to a spherical family of representations

$$\rho: S^m \rightarrow \mathrm{Hom}(G, \mathrm{GL}_n(\mathbb{C}))$$

the K -theory class of the associated vector bundle

$$E_\rho \rightarrow S^m \times BG$$

with holonomy ρ . See Section 6 for the definition of E_ρ , and a more precise statement along these lines (Theorem 6.3). The construction $\rho \mapsto E_\rho$ was considered previously by Ramras–Willett–Yu [39], where it was used to study the (strong) Novikov conjecture, and by Baird–Ramras [7], where it was used to obtain cohomological lower bounds on the homotopy groups of spaces of flat, unitary connections.

The appropriate context for this construction is that of *deformation K -theory*, as studied in [22, 23, 24, 40, 41, 43]. The reduced, unitary deformation K -theory of a group G can be thought of as the homotopy group completion $\Omega B\mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the topological monoid

$$\mathrm{Rep}(G) = \coprod_n \mathrm{Hom}(G, \mathrm{U}(n)),$$

with block sum of matrices as the monoid operation. (A small adjustment is needed to get the correct homotopy group in dimension zero; see Section 4.) General linear deformation K -theory is obtained by replacing $\mathrm{U}(n)$ by $\mathrm{GL}(n) = \mathrm{GL}_n(\mathbb{C})$. The (reduced, unitary) topological Atiyah–Segal map is a homomorphism

$$(1) \quad \tilde{\alpha}_*: \tilde{K}_*^{\mathrm{def}}(G) := \pi_* \Omega B\mathrm{Rep}(G) \longrightarrow \tilde{K}^{-*}(BG),$$

induced by a map of topological monoids arising from the natural map

$$(2) \quad B: \mathrm{Hom}(G, \mathrm{U}(n)) \longrightarrow \mathrm{Map}_*(BG, BU(n)).$$

The details of this construction appear in Section 6.

In order to describe $\tilde{\alpha}_*$ explicitly, we provide a concrete description of the homotopy groups $\tilde{K}_m^{\mathrm{def}}(G)$ in terms of homotopy classes of maps

$$S^m \rightarrow \mathrm{Hom}(G, \mathrm{U}(n)).$$

This description, given in Theorem 4.4, can be viewed as an extension of one of the well-known properties of the homotopy group completion: namely, for each topological monoid M , there is an isomorphism of monoids

$$\pi_0(\Omega BM) \cong \mathrm{Gr}(\pi_0 M),$$

where the left-hand side has the monoid structure induced by loop concatenation, and the right hand side is the Grothendieck group (that is, the ordinary group completion) of the monoid $\pi_0 M$. Theorems 2.8 and 2.17 provide similar descriptions of the higher homotopy groups of ΩBM , under a strong homotopy commutativity condition on M . These results lend some additional justification to the term *deformation K -theory*, since they show that the homotopy groups $\tilde{K}_*^{\mathrm{def}}(G)$ parametrize continuous spherical families (“deformations”) of representations.

Deformation K -theory has proven difficult to compute, and unlike similar functors such as algebraic K -theory of the group ring, or the K -theory of group C^* -algebras, there are no general conjectures describing its behavior. This should be seen as a positive feature of the theory: it is subtle enough to capture delicate information about the group in question, so that when computation can be achieved, concrete consequences follow.

Deformation K -theory was previously used to study the homotopy types of stable moduli spaces of flat connections over surfaces [24, 25, 20, 41]. Our results facilitate such geometric applications by explicitly linking deformation K -theory to spherical families of representations, vector bundles, and spaces of flat connections. Three such applications are provided:

- In Section 7, we reinterpret a result from Ramras–Willet–Yu [39] to show that rational surjectivity of $\tilde{\alpha}_*$ in high dimensions implies the strong Novikov conjecture (Theorem 7.2). Thus surjectivity of $\tilde{\alpha}_*$ should be viewed as a very strong Novikov-type property. We show that surjectivity holds for surface groups (Theorem 7.4), but fails for the 3-dimensional Heisenberg group and for property (T) groups (Section 9).
- In Section 8, we use Lawson’s calculations of deformation K -theory for the 3-dimensional Heisenberg group to produce huge families of homotopy classes in the space of flat, unitary connections on bundles over the Heisenberg manifold. This shows a marked difference between gauge theory in 2- and 3-dimensions: over surfaces, homotopy in the space of flat connections is tightly controlled by Yang–Mills theory and complex geometric considerations, but in 3-dimensions the flood gates open.
- In Section 9, we use results of Tyler Lawson and S. P. Wang to calculate the deformation K -theory of groups G satisfying property (T). By exploiting the fact that our construction of the topological Atiyah–Segal map actually produces a map of E_∞ ring spectra, we deduce that when BG is finite, the vector bundle E_ρ associated to a spherical family of representations always represents a torsion class in $\tilde{K}^0(S^m \times BG)$.

We include a variety of open questions throughout the paper.

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The results in Sections 2, 4, and 6 of this article had a long genesis, beginning with conversations among the author, Willett, and Yu in 2010 surrounding the work in [39]. Weaker versions of the results in these sections were included in a preliminary draft of the author's joint paper with Tom Baird [6], with different arguments. Proposition 8.1 arose from discussions with Robert Lipshitz regarding work of his student Kristen Hendricks [16]. The author thanks Baird, Lipshitz, Willett, and Yu, as well as Markus Szymik and Peter May, for helpful discussions.

NOTATION AND CONVENTIONS

Let (X, x_0) and (Y, y_0) be based spaces.

Given paths $\gamma, \eta: [0, 1] \rightarrow Y$ with $\gamma(1) = \eta(0)$, let $\gamma \square \eta$ denote their concatenation (tracing out γ on $[0, 1/2]$ and η on $[1/2, 1]$). The constant loop at a point $y \in Y$ will be denoted c_y . More generally, the constant map $X \rightarrow \{y\}$ will also be written c_y , or sometimes simply y , when X is clear from context.

We denote the path component of $x \in X$ by $[x]$, and we write $x \simeq x'$ to indicate that $[x] = [x']$. We write $\langle \phi \rangle$ to denote the based homotopy class of a based map $\phi: (X, x_0) \rightarrow (Y, y_0)$, and $[\phi]$ denotes its unbased homotopy class. We write $\phi \simeq \phi'$ to indicate that $[\phi] = [\phi']$. The unbased mapping space will be denoted $\text{Map}(X, Y)$, and the based mapping space will be denoted by $\text{Map}_*(X, Y)$ (when x_0 and y_0 are clear from context). The set of unbased homotopy classes of unbased maps $X \rightarrow Y$ will be denoted $[X, Y]$, and the set of based homotopy classes of based maps will be denoted by $\langle (X, x_0), (Y, y_0) \rangle = \langle X, Y \rangle$.

For $m \geq 1$, we view $\pi_m(X, x_0)$ as the group $\langle (S^m, 1), (X, x_0) \rangle$, with multiplication \square defined via concatenation in the first coordinate of

$$(I^m / \partial I^m, [\partial I^m]) \cong (S^m, 1).$$

For $\phi: (S^m, 1) \rightarrow (X, x_0)$ with $m \geq 1$, we let $\bar{\phi}$ denote the reverse of ϕ with respect to the first coordinate of I^m (so that $\langle \bar{\phi} \rangle = \langle \phi \rangle^{-1}$ in $\pi_m(X, x_0)$).

We set $S^0 = \{\pm 1\}$, with 1 as basepoint. Depending on the situation at hand, we will view $\pi_0(X)$ as either the set of path components of X or as the (naturally isomorphic) set $\langle S^0, X \rangle$.

We will work in the category of compactly generated spaces, which we denote **CGTop**, so in particular all mapping spaces and products have the compactly generated topology associated to the compact-open and product topologies, respectively.

2. THE HOMOTOPY GROUPS OF A HOMOTOPY GROUP COMPLETION

In this section we establish a general result (Theorem 2.8) describing the homotopy groups of the homotopy group completion for certain topological monoids. This result applies in particular to the monoid $\text{Rep}(G)$ defined

in the Introduction, and to its general linear version, for all discrete groups G (see Section 4). Furthermore, the result applies to monoids underlying topological K -theory (Section 5).

We begin by describing the general context for this section. Throughout, M denotes a topological monoid, with monoid operation $(m, n) \mapsto m \bullet n$ (so \bullet is continuous, associative, and there exists a strict identity element in $e \in M$). We define

$$m^k = \underbrace{m \bullet \cdots \bullet m}_k,$$

and by convention $m^0 = e$. All of our results will require M to be homotopy commutative, in the sense that there exists a (possibly unbased) homotopy between \bullet and $\bullet \circ \tau$, where

$$\tau: M \times M \rightarrow M \times M$$

is the twist map $\tau(m, n) = (n, m)$. The *classifying space* of M , denoted BM , is the geometric realization of the topological category \underline{M} with one object $*$ and with morphism space M . Composition in \underline{M} is given by $m \circ n = m \bullet n$. The space BM has a natural basepoint $*$ corresponding to the unique object in \underline{M} . We note that the nerve $N\underline{M}$, which is the simplicial space underlying BM , is the *simplicial bar construction* $B.M$, and has the form $[n] \mapsto M^n$. The *homotopy group completion* of M is the based loop space ΩBM .

There is a natural map

$$\gamma: M \rightarrow \Omega BM,$$

adjoint to the natural map $S^1 \wedge M \rightarrow BM$ (see Section 2.1 for further discussion of this map), and it is a standard fact that γ induces an isomorphism

$$(3) \quad \mathrm{Gr}(\pi_0 M) \xrightarrow{\cong} \pi_0(\Omega BM),$$

where Gr denotes the Grothendieck group.¹

In this section, we present conditions on M under which the higher homotopy groups of ΩBM can be described in a manner analogous to (3). Our description of $\pi_*(\Omega BM)$ will be given in terms of unbased homotopy classes of maps from spheres into M . Given a space X , the space $\mathrm{Map}(X, M)$ becomes a topological monoid, and the set $[X, M]$ of unbased homotopy classes of maps $X \rightarrow M$ becomes a (discrete) monoid. We will often denote the operations in each of these monoids simply by \bullet . We may now form the Grothendieck group $\mathrm{Gr}[X, M]$. If M is homotopy commutative, which we assume from here on, then $[X, M]$ is an abelian monoid (whose monoid operation we write as $+$) and $\mathrm{Gr}[X, M]$ is an abelian group. We will use additive notation $+$ and $-$ when working in $\mathrm{Gr}[X, M]$.

Choosing a basepoint $x \in X$ gives a monoid homomorphism

$$[X, M] \longrightarrow \pi_0(M)$$

¹This can be deduced from the Group Completion Theorem [34] using arguments similar to those in this section.

defined by restriction to x . This homomorphism is split by the homomorphism

$$\pi_0(M) \rightarrow [X, M]$$

sending each path component to the homotopy class of a constant map into that component. Hence $[X, M]$ contains a copy of $\pi_0(M)$ (generated by nullhomotopic maps), and it follows that $\mathrm{Gr}[X, M]$ contains a copy of $\mathrm{Gr}(\pi_0 M)$ as a direct summand (consisting of formal differences between nullhomotopic maps).

Definition 2.1. For $k \in \mathbb{N}$, let $\Pi_k(M)$ denote the Grothendieck group $\mathrm{Gr}[S^k, M]$, and define

$$\tilde{\Pi}_k(M) = \Pi_k(M) / \mathrm{Gr}(\pi_0 M).$$

Note that we have natural direct sum decompositions

$$\Pi_k(M) \cong \tilde{\Pi}_k(M) \oplus \mathrm{Gr}(\pi_0 M),$$

with $\tilde{\Pi}_k(M)$ corresponding to the subgroup consisting of those formal differences $[\phi] - [\psi]$ for which $\phi(1)$ and $\psi(1)$ lie in the same path component of M . Note also that there is a natural isomorphism $\tilde{\Pi}_0(M) \cong \mathrm{Gr}(\pi_0 M)$, induced by sending the class represented by $f: S^0 \rightarrow M$ to $[f(-1)] - [f(1)]$.

In Section 2.1, we will construct a natural map

$$\tilde{\Gamma}: \tilde{\Pi}_k(M) \longrightarrow \pi_k(\Omega B M),$$

and we will show that $\tilde{\Gamma}$ is an isomorphism under certain conditions on M . We now explain these conditions.

Definition 2.2. We say that a topological monoid M is *proper* if the inclusion of the identity element is a closed cofibration.

Remark 2.3. If M is proper, then the degeneracy maps for the simplicial space $N.\underline{M}$ underlying BM are all closed cofibrations. This implies that the natural map from the “thick” geometric realization of $N.\underline{M}$ to its “thin” realization (namely BM) is a homotopy equivalence [47]. We will work with the thin realization throughout, but it should be noted that the thick realization is used in [34], and the results from that paper play a key role in our arguments. Additionally, it follows from Lillig’s Union Theorem ([26] or [48, Chapter 5]) that $N.\underline{M}$ is *proper* in the sense of [28, Appendix]. If a simplicial map between proper simplicial spaces is a homotopy equivalence on each level, then the same is true for the induced map between realizations [28, Appendix]. In fact, the same statement holds for weak equivalences in place of homotopy equivalences [36]. These results will be needed in the proof of Theorem 2.8.

We will need to consider the action of $\pi_1(M, m)$ on $\pi_k(M, m)$. We use the conventions in Hatcher [15, Section 4.1], so that this is a *left* action. We note that if $[m]$ is invertible in $\pi_0(M)$, then this action is trivial (this

follows from [15, Example 4A.3], for instance, which shows that the identity component of an H -space is always simple).

Definition 2.4. Consider a topological monoid M and a natural number $k \geq 1$. Given $m \in M$, let $[m] \subset M$ be its path component, viewed as a subspace of M . We say that m is *k-anchored* there exists a homotopy

$$H: [m] \times [m] \times I \rightarrow [m^2]$$

such that $H_0 = \bullet$, $H_1 = \bullet \circ \tau$, and the loop $\eta(t) = H(m, m, t)$ acts trivially on $\pi_k(M, m^2)$. When k is clear from context, we will refer to H as a *homotopy anchoring* m . We say that m is *strongly k-anchored* if there are infinitely many $n \in \mathbb{N}$ for which m^n is k -anchored.

We say that m is (strongly) *anchored* if it is (strongly) k -anchored for all $k \geq 1$. We say that a path component C of M is (strongly) k -anchored (or anchored) if there exists an element $m \in C$ that is (strongly) k -anchored (respectively, anchored).

Remark 2.5. It is an elementary exercise to check that if $m_0 \simeq m_1$ in M , then m_0 is (strongly) k -anchored if and only if m_1 is (strongly) k -anchored.

Examples 2.6. If M is (strictly) abelian, then every element of M is strongly anchored, since we can take H to be the constant homotopy.

If M is homotopy commutative, then every path component of M with abelian fundamental group is 1-anchored (since the action of π_1 on itself is conjugation). If M is homotopy commutative and every path component of M is a simple space (e.g. if $\pi_0(M)$ is a group), then every element in M is strongly anchored.

We will see more interesting examples in Sections 3, 4, and 5.

Remark 2.7. In [40], an element $m_0 \in M$ is called anchored if all powers of m_0 are anchored and the loops η described in Definition 2.4 are all *constant*. A small modification to the proof of [40, Lemma 3.13] shows that the results in that article hold if one simply requires m_0 to be strongly 1-anchored in the sense defined above (see also [40, Remark 3.7]). Following the notation of that paper, let $\tilde{\alpha}$ and $\tilde{\beta}$ be loops in M based at $m \in M$. Let $\tilde{\alpha} \bullet \tilde{\beta}$ denote the pointwise product of these loops (so that $(\tilde{\alpha} \bullet \tilde{\beta})(t) = \tilde{\alpha}(t) \bullet \tilde{\beta}(t)$), and similarly for $\tilde{\beta} \bullet \tilde{\alpha}$.

The aim of [40, Lemma 3.13] is, essentially, to show that $\langle \tilde{\alpha} \bullet \tilde{\beta} \rangle = \langle \tilde{\beta} \bullet \tilde{\alpha} \rangle$ whenever there exists a homotopy H anchoring m . We now explain how to modify the argument from [40] to work whenever $\langle \eta \rangle$ is central in $\pi_1(M, m^2)$.

For each $s \in [0, 1]$, let η_s be the path $\eta_s(t) = \eta(st)$, and set

$$h_s(t) = H(\tilde{\alpha}(t), \tilde{\beta}(t), s),$$

so that h_s is a loop based at $\eta(s) = \eta_s(1)$.

The homotopy of loops $s \mapsto h_s$ is used in the proof of [40, Lemma 3.13]. To extend that argument to the present context, one can replace h_s by

$$g_s = \eta_s \square h_s \square \overline{\eta_s}.$$

Note that for each $s \in [0, 1]$, g_s is a loop based at m^2 , and $\langle g_0 \rangle = \langle \tilde{\alpha} \bullet \tilde{\beta} \rangle$. Also, $g_1 = \eta \square (\tilde{\beta} \bullet \tilde{\alpha}) \square \bar{\eta}$, so centrality of $\langle \eta \rangle$ implies that $\langle g_1 \rangle = \langle \tilde{\beta} \bullet \tilde{\alpha} \rangle$. Thus $\langle \tilde{\alpha} \bullet \tilde{\beta} \rangle = \langle \tilde{\beta} \bullet \tilde{\alpha} \rangle$, as desired. For further details, compare with [40].

In order to motivate the construction of the map $\tilde{\Gamma}$ in Section 2.1, we now state the main result of this section. Recall that a subset S of a monoid N is called *cofinal* if for each $n \in N$ there exists $n' \in N$ such that $n \bullet n' \in S$.

Theorem 2.8. *Let M be a proper, homotopy commutative topological monoid such that the subset of strongly 1-anchored components is cofinal in $\pi_0(M)$. Then for each $k \geq 0$, the natural map*

$$\tilde{\Gamma}: \tilde{\Pi}_k(M) \longrightarrow \pi_k(\Omega BM)$$

is an isomorphism.

We will see examples of monoids to which this result applies in Sections 3, 4, and 5.

Remark 2.9. The isomorphism in Theorem 2.8 *does not* hold for all homotopy commutative topological monoids. For instance, let

$$M = \coprod_P B\text{Aut}(P),$$

where R is a ring and P runs over a set of representatives for the isomorphism classes of finitely generated projective R -modules. Direct sum makes M a homotopy commutative topological monoid, and the homotopy groups $\pi_*(\Omega BM) \cong K_*(R)$ are the algebraic K -theory groups of the ring R [34]. However, $B\text{Aut}(P)$ is the classifying space of the discrete group $\text{Aut}(P)$, so for $k \geq 2$ we have $\pi_k(B\text{Aut}(P)) = 0$ and hence every map $S^k \rightarrow M$ is nullhomotopic. Thus $\tilde{\Pi}_k(M) = 0$ for $k \geq 2$, whereas $K_*(R)$ is in general quite complicated.

We end this section by establishing a helpful universal property of the natural map

$$(4) \quad [S^k, M] \xrightarrow{i} \text{Gr}[S^k, M] \xrightarrow{q} \tilde{\Pi}_k(M),$$

where i is the universal map from the monoid $[S^k, M]$ to its group completion, and q is the quotient map. We denote the composite (4) by π .

Proposition 2.10. *Let M be a homotopy commutative topological monoid in which the subset of k -anchored components is cofinal. Then the map $\pi: [S^k, M] \rightarrow \tilde{\Pi}_k(M)$ is surjective, and if $f: [S^k, M] \rightarrow P$ is a monoid homomorphism that sends all nullhomotopic maps to the identity, then f factors uniquely as $\bar{f} \circ \pi$.*

Furthermore $\pi([\phi]) = 0$ if and only if $[\phi]$ is stably nullhomotopic in the sense that there exists a constant map $c: S^k \rightarrow M$ such that $\phi \bullet c$ is nullhomotopic.

In other words, $\widetilde{\Pi}_k(M)$ is the quotient, in the category of monoids, of $[S^k, M]$ by the submonoid $S \subset [S^k, M]$ of stably nullhomotopic maps. It follows that the submonoid of stably nullhomotopic maps is the normal closure of the submonoid N of nullhomotopic maps, in the sense that S is the smallest submonoid containing N that is the kernel of a monoid homomorphism.

For the proof of Proposition 2.10, we need a version of the Eckmann–Hilton argument, and first we record a basic fact regarding the action of the fundamental group on higher homotopy.

Lemma 2.11. *Consider a (not necessarily based) homotopy α_s of maps $S^k \rightarrow X$ ($k \geq 1$), and let $\eta(t) = \alpha_t(1)$ be the track of this homotopy on the basepoint $1 \in S^k$. Then $\langle \alpha_0 \rangle = \langle \eta \rangle \cdot \langle \alpha_1 \rangle$ in $\pi_k(X, \alpha_0(1))$.*

Proof. In general, the action of $\pi_1(X, x_0)$ on $\pi_k(X, x_0)$ is induced by an operation which takes in a map $\gamma: [0, s] \rightarrow X$ (for some $s \in [0, 1]$) and a map $\alpha: (I^k, \partial I^k) \rightarrow (X, \gamma(s))$ and produces a map

$$\gamma \cdot \alpha: (S^k, 1) \rightarrow (X, \gamma(0))$$

defined by shrinking the domain of α to a concentric cube $C \subset I^k$ of side length $1 - s/2$ and filling in the path γ on each radial segment connecting ∂C to ∂I^k (compare with Hatcher [15, Section 4.1], for instance). In this language, the desired homotopy is simply $s \mapsto \eta|_{[0, s]} \cdot \alpha_s$. \square

Lemma 2.12. *Let M be a topological monoid and let $m \in M$ be k -anchored. Then for any $\phi, \psi: S^k \rightarrow M$ ($k \geq 1$) with $\phi(1) = \psi(1) = m$ we have*

$$\phi \bullet \psi \simeq (\phi \bullet c_m) \square (\psi \bullet c_m) = (\phi \square \psi) \bullet c_m.$$

In particular, setting $\psi = \overline{\phi}$ gives

$$\phi \bullet \overline{\phi} \simeq \overline{\phi} \bullet \phi \simeq c_{m^2}.$$

Proof. Just as in the ordinary Eckmann–Hilton argument, the point is that \bullet is a homomorphism

$$\pi_k(M, m) \times \pi_k(M, m) \xrightarrow{\bullet} \pi_k(M, m^2).$$

The relevant equation holds on the nose, not just up to homotopy: for all maps $\alpha, \beta, \alpha', \beta': S^k \rightarrow M$ satisfying $\alpha(1) = \beta(1)$ and $\alpha'(1) = \beta'(1)$,

$$(\alpha \square \beta) \bullet (\alpha' \square \beta') = (\alpha \bullet \alpha') \square (\beta \bullet \beta').$$

Hence we have:

$$\phi \bullet \psi \simeq (\phi \square c_m) \bullet (c_m \square \psi) = (\phi \bullet c_m) \square (c_m \bullet \psi).$$

To complete the proof, it suffices to show that $\langle c_m \bullet \psi \rangle = \langle \psi \bullet c_m \rangle$. Let H be a homotopy anchoring m , and set $\eta(t) = H(m, m, t)$. By Lemma 2.11,

$$\langle c_m \bullet \psi \rangle = \langle \eta \rangle \cdot \langle \psi \bullet c_m \rangle,$$

and since H anchors m , we have $\langle \eta \rangle \cdot \langle \psi \bullet c_m \rangle = \langle \psi \bullet c_m \rangle$. \square

Proof of Proposition 2.10. Each element in $\text{Gr}[S^k, M]$ has the form $[\phi] - [\psi]$ for some $\phi, \psi: S^k \rightarrow M$. By assumption, there exists $m \in M$ such that $\psi(1) \bullet m$ is k -anchored. Adding $[c_m]$ to both $[\phi]$ and $[\psi]$ if necessary, we may assume that $\psi(1)$ is k -anchored.

By Lemma 2.12, $\psi \bullet \bar{\psi}$ is nullhomotopic, so the element

$$[\phi] - [\psi] = [\phi \bullet \bar{\psi}] - [\psi \bullet \bar{\psi}] \in \text{Gr}[S^k, M]$$

is equivalent, modulo $\text{Gr}(\pi_0 M)$, to $[\phi \bullet \bar{\psi}]$, which is in the image of π . Hence π is surjective.

Now say $f: [S^k, M] \rightarrow P$ is a homomorphism sending all nullhomotopic maps to the identity. Since π is surjective, \bar{f} is completely determined by the equation $\bar{f}(\pi[\phi]) = f([\phi])$. To prove that \bar{f} is well-defined, say $\pi([\phi]) = \pi([\psi])$ for some $\phi, \psi: S^k \rightarrow M$. Then there exist $x, y \in M$ such that

$$[\phi] - [\psi] = [c_x] - [c_y],$$

in $\text{Gr}[S^k, M]$, and hence there exists $\tau: S^k \rightarrow M$ such that

$$[\phi] + [c_y] + [\tau] = [c_x] + [\psi] + [\tau]$$

in $[S^k, M]$. Again, we may assume without loss of generality that $\tau(1)$ is k -anchored. Adding $[\bar{\tau}]$ to both sides and applying f , we have

$$(5) \quad f([\phi]) + f([c_y]) + f([\tau \bullet \bar{\tau}]) = f([c_x]) + f([\psi]) + f([\tau \bullet \bar{\tau}]).$$

By Lemma 2.12, $\tau \bullet \bar{\tau}$ is nullhomotopic. Since f sends all nullhomotopic maps to the identity, Equation (5) reduces to $f([\phi]) = f([\psi])$, showing that \bar{f} is well-defined. It follows from the equation $f = \bar{f} \circ \pi$ (together with surjectivity of π) that \bar{f} is a homomorphism as well.

Finally, say $\pi([\phi]) = 0$ for some $[\phi] \in [S^k, M]$. Then there exist constant maps $a, b: S^k \rightarrow M$, such that

$$[\phi] = [a] - [b]$$

in $\text{Gr}[S^k, M]$. This means that

$$\phi \bullet b \bullet \psi \simeq a \bullet \psi$$

for some $\psi: S^k \rightarrow M$, and we may assume that $\psi(1)$ is k -anchored. Multiplying both sides by $\bar{\psi}$ and applying Lemma 2.12 gives

$$\phi \bullet b \bullet \psi(1)^2 \simeq a \bullet \psi(1)^2,$$

where $\psi(1)^2$ denotes the constant map with image $\psi(1)^2$. The right-hand side is constant, as is $b \bullet \psi(1)^2$, so ϕ is stably nullhomotopic, as desired. \square

2.1. Construction of the map $\tilde{\Gamma}$. We now give the details behind the construction of the natural map

$$\tilde{\Gamma}: \tilde{\Pi}_k(M) = \text{Gr}[S^k, M] / \text{Gr}(\pi_0 M) \longrightarrow \pi_k(\Omega BM).$$

We begin by discussing an alternate model for the groups $\pi_k(\Omega BM)$. Concatenation of loops makes ΩBM into an H -space with the constant

loop c_* as identity, so the identity component of ΩBM is simple. It now follows from Lemma 2.11 (see also [15, Section 4.A]) that the natural map

$$(6) \quad J: \pi_k \Omega BM = \langle (S^k, 1), (\Omega BM, c_*) \rangle \longrightarrow [S^k, \Omega BM]_*$$

is bijective for each $k \geq 0$, where right-hand side is the set of unbased homotopy classes of maps $f: S^k \rightarrow \Omega BM$ such that $f(1)$ is homotopic to the constant loop c_* .

There is a natural operation on $[S^k, \Omega BM]_*$ coming from the H -space structure of ΩBM . This operation is induced by the pointwise concatenation map

$$(7) \quad \square: \text{Map}(S^k, \Omega BM) \times \text{Map}(S^k, \Omega BM) \longrightarrow \text{Map}(S^k, \Omega BM),$$

defined by

$$(\alpha \square \beta)(z) = \alpha(z) \square \beta(z).$$

The map \square also induces an operation on $\langle S^k, \Omega BM \rangle = \pi_k \Omega BM$, and the bijection (6) is a homomorphism with respect to these operations. Moreover, when $k \geq 1$, the Eckmann-Hilton argument [12] shows that the operation on $\langle S^k, \Omega BM \rangle$ induced by \square agrees with the usual multiplication operation on $\pi_k \Omega BM$. When $k = 0$, we give $\pi_0 \Omega BM$ the monoid structure induced by \square . With this understood, we may now view J as a group isomorphism.

Recall that there is a natural map

$$(8) \quad [0, 1] \times M \rightarrow BM$$

resulting from the fact that the simplicial space underlying BM has M as its space of 1-simplices². Since BM has a single 0-simplex, and since the 1-simplex corresponding to the identity element $e \in M$ is degenerate, the map (8) descends to a map

$$S^1 \wedge M \longrightarrow BM$$

whose adjoint will be denoted by

$$(9) \quad \gamma: M \longrightarrow \Omega BM.$$

We note that γ is natural with respect to continuous homomorphisms of topological monoids. One might like to define a map

$$[S^k, M] \longrightarrow [S^k, \Omega BM]_* \cong \pi_k(\Omega BM)$$

via composition with γ , but some correction is needed to make this map land in $[S^k, \Omega BM]_*$.

Given $\alpha \in \Omega BM$ and $g: S^k \rightarrow \Omega BM$, we simplify notation by writing $\alpha \square g$ in place of $c_\alpha \square g$. For each $k \geq 0$, we now define

$$\begin{aligned} \Gamma: [S^k, M] &\longrightarrow [S^k, \Omega BM] \\ [f] &\mapsto [\overline{\gamma(f(1))} \square (\gamma \circ f)]. \end{aligned}$$

²Our conventions on geometric realization come from Milnor [35], and η is induced by the homeomorphism $I = [0, 1] \rightarrow \Delta^1 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 \mid 0 = t_0 \leq t_1 \leq t_2 = 1\}$ given by $t \mapsto (0, t, 1)$.

We note that there is a potential ambiguity in this notation: the symbol $\overline{\gamma(f(1))}$ refers to the constant map with image $\overline{\gamma(f(1))} \in \Omega BM$, not to the reverse of the constant map with image $\gamma(f(1))$ (of course a constant map is its own reverse). We will continue to use this notation throughout the section.

It is straightforward to check that Γ is well-defined on unbased homotopy classes, and for every $[f]$, we have $\Gamma([f]) \in [S^k, \Omega BM]_*$ since evaluating at $1 \in S^k$ gives the loop $\overline{\gamma(f(1))} \square \gamma(f(1)) \simeq c_*$.

Proposition 2.13. *For each $k \geq 0$, the function Γ is a monoid homomorphism, natural in M , and induces a natural homomorphism*

$$\overline{\Gamma}_M: \widetilde{\Pi}_k(M) \longrightarrow [S^k, \Omega BM]_*$$

Definition 2.14. Let M be a homotopy commutative topological monoid. Then we define

$$\tilde{\Gamma} = \tilde{\Gamma}_M := J^{-1} \circ \overline{\Gamma}_M: \widetilde{\Pi}_k(M) \longrightarrow \pi_k \Omega BM,$$

where J is the (natural) isomorphism (6).

The proof of Proposition 2.13 will use the following elementary lemma.

Lemma 2.15. *Let (M, \bullet) be a topological monoid. Then the diagram*

$$(10) \quad \begin{array}{ccc} M \times M & \xrightarrow{\bullet} & M \\ \downarrow \gamma \times \gamma & & \downarrow \gamma \\ \Omega BM \times \Omega BM & \xrightarrow{\square} & \Omega BM \end{array}$$

is homotopy commutative. Moreover, if M is homotopy commutative then the maps

$$M \times M \rightarrow \Omega BM$$

given by $(m, n) \mapsto \gamma(m) \square \gamma(n)$ and $(m, n) \mapsto \gamma(n) \square \gamma(m)$ are homotopic.

Proof. The space of 2-simplices in the simplicial space BM homeomorphic to $M \times M$, with (m, n) corresponding to the sequence of composable morphisms

$$* \xrightarrow{(n, *)} * \xrightarrow{(m, *)} *.$$

We describe the desired homotopy $M \times M \times I \rightarrow \Omega BM$ by specifying its adjoint, which is induced by a map of the form

$$(M \times M \times I) \times I = M \times M \times (I \times I) \xrightarrow{\text{Id}_M \times \text{Id}_M \times H} M \times M \times \Delta^2 \xrightarrow{\pi} BM,$$

where $H: I \times I \rightarrow \Delta^2$ is defined below and π is induced by the definition of geometric realization. Set

$$\Delta^2 = \{(t_1, t_2) \in I \times I : t_1 \leq t_2\},$$

and define

$$\vec{w}_t = (1-t)(0, 1) + t(1/2, 1/2) = (t/2, 1-t/2) \in \Delta^2.$$

The map H is defined by

$$H(t, s) = \begin{cases} 2s\vec{w}_t & \text{if } 0 \leq s \leq 1/2 \\ (2s-1)(1, 1) + (2-2s)\vec{w}_t & \text{if } 1/2 \leq s \leq 1, \end{cases}$$

and one may check that it has the desired properties (note that we are using the conventions regarding (co)face and (co)degeneracy maps from [35]). This proves commutativity of (10).

When M is homotopy commutative, the second statement in the lemma follows from the first: we have

$$\square \circ (\gamma \times \gamma) \simeq \gamma \circ \bullet \simeq \gamma \circ \bullet \circ \tau \simeq \square \circ (\gamma \times \gamma) \circ \tau.$$

□

Proof of Proposition 2.13. First we show that Γ is a monoid homomorphism. Given $\phi, \psi: S^k \rightarrow M$, we must show that

$$\Gamma([\phi \bullet \psi]) = \Gamma([\phi]) \square \Gamma([\psi]),$$

or in other words that

$$\overline{\gamma(\phi(1) \bullet \psi(1))} \square (\gamma \circ (\phi \bullet \psi)) \simeq \overline{\gamma(\phi(1))} \square \gamma \circ \phi \square \overline{\gamma(\psi(1))} \square \gamma \circ \psi.$$

Applying Lemma 2.15 gives

$$\begin{aligned} \overline{\gamma(\phi(1) \bullet \psi(1))} \square (\gamma \circ (\phi \bullet \psi)) &\simeq \left(\overline{\gamma(\phi(1))} \square \gamma(\psi(1)) \right) \square (\gamma \circ \phi \square \gamma \circ \psi) \\ &= \left(\overline{\gamma(\psi(1))} \square \overline{\gamma(\phi(1))} \right) \square (\gamma \circ \phi \square \gamma \circ \psi) \end{aligned}$$

Since the operation \square is homotopy associative, to complete the proof that Γ is a homomorphism it remains only to show that $\overline{\gamma(\psi(1))}$, $\overline{\gamma(\phi(1))}$, and $[\gamma \circ \phi]$ commute with one another under the operation \square . By Lemma 2.15, $[\gamma(\psi(1))]$, $[\gamma(\phi(1))]$, and $[\gamma \circ \phi]$ commute with one another, which suffices because $[\gamma(\psi(1))]$ and $[\gamma(\phi(1))]$ are the inverses of $\overline{\gamma(\psi(1))}$ and $\overline{\gamma(\phi(1))}$ (respectively) under \square .

It follows from the definitions that Γ sends all nullhomotopic maps to the identity element in $[S^k, \Omega BM]_*$, so Proposition 2.10 implies that Γ induces a group homomorphism

$$\overline{\Gamma}: \widetilde{\Pi}_k(M) = \Pi(M)/\text{Gr}(\pi_0 M) \longrightarrow [S^k, \Omega BM]_*$$

as desired. Naturality of Γ , and hence of $\overline{\Gamma}$, follows from naturality of γ . □

2.2. Stably group-like monoids. We now show that under certain conditions, it is possible to construct an inverse to the homomorphism

$$\widetilde{\Gamma}: \widetilde{\Pi}_k(M) \longrightarrow \pi_k(\Omega BM)$$

introduced in Section 2.1. We recall some terminology from Ramras [40, Section 3].

Definition 2.16. A topological monoid M is *stably group-like* with respect to an element $[m] \in \pi_0(M)$ if the submonoid of $\pi_0(M)$ generated by $[m]$ is cofinal. More explicitly, M is stably group-like with respect to m if for every $x \in M$, there exists $x' \in M$ and $k \geq 0$ such that $x \bullet x'$ lies in the same path component as m^k .

Given $m_0 \in M$, we write $M \xrightarrow{\bullet m_0} M$ to denote the map $m \mapsto m \bullet m_0$. We define

$$M_\infty(m_0) = \text{telescope} \left(M \xrightarrow{\bullet m_0} M \xrightarrow{\bullet m_0} M \xrightarrow{\bullet m_0} \cdots \right),$$

where the right-hand side is the infinite mapping telescope of this sequence. As in [40], we write points in $M_\infty(m_0)$ as equivalence classes of triples (m, n, t) , where $m \in M$, $n \in \mathbb{N}$, $t \in [0, 1]$, and

$$(m, n, 1) \sim (m \bullet m_0, n + 1, 0)$$

for each $n \in \mathbb{N}$. We always use $[(e, 0, 0)] \in M_\infty(m_0)$ as the basepoint.

Our next goal is to prove the following special case of Theorem 2.8.

Proposition 2.17. *If M is a proper, homotopy commutative topological monoid that is stably group-like with respect to a strongly 1-anchored element $m_0 \in M$, then the natural map*

$$\tilde{\Gamma}: \tilde{\Pi}_k(M) \longrightarrow \pi_k(\Omega BM)$$

is an isomorphism for each $k \geq 0$.

Before giving the proof, we need to review some facts surrounding the Group Completion Theorem [34], which provides an isomorphism

$$(11) \quad \pi_k(\Omega BM) \xrightarrow{\cong} \pi_k(M_\infty(m_0))$$

under the conditions in Proposition 2.17. We will give an explicit description of this isomorphism in Lemma 2.18 below. The proof of Proposition 2.17 will then proceed by constructing another map

$$\Psi: \pi_k(M_\infty(m_0)) \longrightarrow \tilde{\Pi}_k(M)$$

so that the composite

$$\pi_k(\Omega BM) \xrightarrow{\cong} \pi_k(M_\infty(m_0)) \xrightarrow{\Psi} \tilde{\Pi}_k(M)$$

is inverse to $\tilde{\Gamma}$.

If M is proper and stably group-like with respect to a strongly 1-anchored component $[m_0]$, then (11) is induced by a zig-zag of weak equivalences, as we now explain. The monoid M acts continuously on $M_\infty(m_0)$ via

$$m \cdot [(m', n, t)] = [(m \bullet m', n, t)].$$

An action of M on a space X gives rise to a category (internal to **CGTop**) with object space X and morphism space $M \times X$; the morphism (m, x) has domain x and range $m \cdot x$, and composition is just multiplication in M : $(n, m \cdot x) \circ (m, x) = (n \bullet m, x)$. We denote the classifying space of this

category by X_M . Since $\{*\}_M \cong BM$ (where $\{*\}$ is the one-point space), we get a canonical map

$$q: X_M \rightarrow BM$$

induced by the projection $X \rightarrow \{*\}$. When $X = M_\infty(m_0)$, we call this projection map $q(M, m_0)$.

The isomorphism (11) is induced by a zig-zag of weak equivalences of the form

$$(12) \quad \Omega BM \xrightarrow{\simeq} \text{hofib}(q(M, m_0)) \xleftarrow{\simeq} M_\infty(m_0).$$

Here $\text{hofib}(q(M, m_0))$ is the homotopy fiber of $q(M, m_0)$ over the basepoint $* \in BM$. Points in $\text{hofib}(q(M, m_0))$ are pairs

$$(z, \beta) \in (M_\infty(m_0))_M \times \text{Map}(I, BM)$$

with $\beta(0) = q(M, m_0)(z)$ and $\beta(1) = *$. The basepoint of $\text{hofib}(q(M, m_0))$ is the pair $[(e, 0, 0), c_*]$, where $[(e, 0, 0)] \in (M_\infty(m_0))_M$ corresponds to the point $[(e, 0, 0)]$ in the object space $M_\infty(m_0)$ of the category underlying $(M_\infty(m_0))_M$.

The first map in (12) is induced by sending a based loop $\alpha: S^1 \rightarrow BM$ to the point $[(e, 0, 0), \alpha] \in \text{hofib}(q(M, m_0))$. It is a weak equivalence because $(M_\infty(m_0))_M$ is weakly contractible (see [34, p. 281] or [40, pp. 2251–2252]). Note here that $\Omega BM \cong \text{hofib}(* \rightarrow BM)$.

The second map in (12) is the natural inclusion of the fiber of $q(M, m_0)$ over $* \in BM$ into the homotopy fiber. The fact that this map is a weak equivalence is established in [40, Proof of Theorem 3.6]. The main step in the argument is to show that the fundamental group of $M_\infty(M)$ is abelian for *all* choices of basepoint. This part of the argument is the only place in the proofs of Theorems 2.8 and 2.17 where we need m_0 to be strongly 1-anchored (see Remark 2.7 regarding the difference between the notion of anchored in the present paper and the notion used in [40]).

For each $k \geq 0$, there is a natural isomorphism

$$(13) \quad \pi_k(M_\infty(m_0)) \cong \text{colim} \left(\pi_k(M, e) \xrightarrow{\bullet m_0} \pi_k(M, m_0) \xrightarrow{\bullet m_0} \dots \right),$$

where the maps in the colimit on the right are those induced by (right) multiplication by the constant map c_{m_0} . We will denote the colimit on the right by

$$(14) \quad \text{colim}_{n \rightarrow \infty} \pi_k(M, m_0^n).$$

Let $i_n: M \rightarrow M_\infty(m_0)$ denote the inclusion of M into the n th stage of the mapping telescope; explicitly

$$(15) \quad \begin{aligned} M &\xrightarrow{i_n} M_\infty(m_0) \\ m &\longmapsto [(m, n, 0)]. \end{aligned}$$

Additionally, define

$$(16) \quad \begin{aligned} M &\xrightarrow{f_n} \Omega BM \\ m &\longmapsto \overline{\gamma(m_0^n)} \square \gamma(m), \end{aligned}$$

where $\gamma: M \rightarrow \Omega BM$ is the map (9).

Lemma 2.18. *Let M and m_0 be as in Proposition 2.17. Then for each element $\alpha \in \pi_k(M, m_0^n)$, the isomorphism*

$$\pi_k(\Omega BM) \xrightarrow{\cong} \pi_k(M_\infty(m_0))$$

induced by the zig-zag (12) carries $(f_n)_(\alpha)$ to $(i_n)_*(\alpha)$.*

Note that every class in $\pi_k(M_\infty(m_0))$ has the form $(i_n)_*(\alpha)$ for some $\alpha \in \pi_k(M, m_0^n)$ (by (13)), so Lemma 2.18 completely determines the isomorphism

$$\pi_k(\Omega BM) \xrightarrow{\cong} \pi_k(M_\infty(m_0))$$

induced by (12). By abuse of notation, we will denote this isomorphism by $i \circ f^{-1}$ from now on.

Proof of Lemma 2.18. It suffices to show that the diagram

$$(17) \quad \begin{array}{ccccc} & & M & & \\ & f_n \swarrow & & \searrow i_n & \\ \Omega BM & \xrightarrow{\cong} & \text{hofib}(q(M, m_0)) & \xleftarrow{\cong} & M_\infty(m_0), \end{array}$$

is homotopy commutative. This can be proven using the argument at the end of the proof of [40, Theorem 3.6]. That argument shows that the diagram commutes after passing to connected components, and it is routine to check that the paths constructed there give rise to a continuous homotopy. \square

Remark 2.19. When $k = 0$, the colimit (14) has a monoid structure defined as follows: denoting elements in $\pi_k(M, m_0^n)$ ($n = 0, 1, 2, \dots$) by pairs $([m], n)$ with $m \in M$, we define

$$([m], n) + ([m'], n') = ([m \bullet m'], n + n').$$

This monoid structure is in fact a group structure since M is stably group-like with respect to $[m_0]$, and the bijection

$$\pi_0(\Omega BM) \cong \operatorname{colim}_{n \rightarrow \infty} \pi_0(M, m_0^n)$$

given by composing (11) and (13) is a monoid isomorphism (see Ramras [40, Theorem 3.6] for details).

Proof of Proposition 2.17. Given a based homotopy class $\langle \phi \rangle \in \pi_k(M, m_0^n)$, we define

$$\Psi_n(\langle \phi \rangle) = [\phi] \in \tilde{\Pi}_k(M).$$

Since all constant maps are trivial in $\tilde{\Pi}_k(M)$, the maps Ψ_n are compatible with the structure maps for $\operatorname{colim}_{n \rightarrow \infty} \pi_k(M, m_0^n)$ and induce a well-defined function

$$\Psi: \operatorname{colim}_{n \rightarrow \infty} \pi_k(M, m_0^n) \rightarrow \tilde{\Pi}_k(M).$$

Let $\Phi = \Psi \circ (i \circ f^{-1})$, where $i \circ f^{-1}$ is the map from Lemma 2.18. We will show that $\tilde{\Gamma}$ and Φ are inverses of one another.³

First, consider $\tilde{\Gamma} \circ \Phi$. As noted above, each element of $\pi_k(\Omega BM)$ has the form $(f_n)_* \langle \phi \rangle$ for some $\langle \phi \rangle \in \pi_k(M, m_0^n)$. Now

$$\begin{aligned} \tilde{\Gamma} \circ \Phi((f_n)_* \langle \phi \rangle) &= \tilde{\Gamma} \circ \Psi((i_n)_* \langle \phi \rangle) = \tilde{\Gamma}([\phi]) \\ &= J^{-1} \left(\overline{[\gamma(m_0^n)]} \sqcup \gamma \circ \phi \right) = (f_n)_* \langle \phi \rangle, \end{aligned}$$

so $\tilde{\Gamma} \circ \Phi$ is the identity map.

Next, consider the composition $\Phi \circ \tilde{\Gamma}$. The group $\tilde{\Pi}_k(M)$ is generated by classes of the form $[\phi]$ with $\phi: S^k \rightarrow M$, so it will suffice to check that $\Phi \circ \tilde{\Gamma}([\phi]) = [\phi]$ for each $\phi: S^k \rightarrow M$. Since M is stably group-like with respect to $[m_0]$, there exists $m \in M$ such that $\phi(1) \bullet m$ lies in the path component of m_0^n (for some n). The maps ϕ and $\phi \bullet c_m$ represent the same class in $\tilde{\Pi}_k(M)$, so we may assume without loss of generality that $[\phi(1)] = [m_0^n]$, and in fact we may assume $\phi(1) = m_0^n$ since the basepoint $1 \in S^k$ is non-degenerate. Now

$$\tilde{\Gamma}([\phi]) = J^{-1} \left(\overline{[\gamma(\phi(1))]} \sqcup (\gamma \circ \phi) \right) = J^{-1} \left(\overline{[\gamma(m_0^n)]} \sqcup (\gamma \circ \phi) \right) = (f_n)_* \langle \phi \rangle.$$

Applying $(i \circ f^{-1})$ to this element gives $(i_n)_* \langle \phi \rangle$, which maps to $[\phi]$ under Ψ as desired. \square

2.3. Proof of Theorem 2.8. We will need a definition.

Definition 2.20. Given a topological monoid M and a submonoid $N \subset \pi_0(M)$, we define

$$(18) \quad \overline{N} = \{m \in M \mid [m] \in N\}.$$

More generally, if S is an arbitrary subset of $\pi_0(M)$, we define

$$\overline{S} = \langle \overline{S} \rangle,$$

where $\langle S \rangle$ is the submonoid of $\pi_0(M)$ generated by S . Note that \overline{S} is a submonoid of M , and $\pi_0(\overline{S}) = \langle S \rangle$. Moreover, \overline{S} is a union of path components of M .

Observe that if M is homotopy commutative, so is \overline{S} (for every $S \subset \pi_0(M)$), and if $m \in \overline{S}$ is strongly 1-anchored in M , the same is true in \overline{S} .

³It then follows that Φ is a homomorphism, and that it is independent of a_0 and is natural. Since $i \circ f^{-1}$ is a homomorphism (in fact, an isomorphism), we also conclude that Ψ is a homomorphism (and in fact, an isomorphism).

Proof of Theorem 2.8. Consider the set \mathcal{F} of all finite subsets of $\pi_0(M)$, which forms a directed poset under inclusion. For each $F \in \mathcal{F}$, let $\sigma(F) \in \pi_0(M)$ be the product of all elements in F (this is well-defined, since M is homotopy commutative). Since the subset of strongly 1-anchored components is cofinal in $\pi_0(M)$, the set

$$\mathcal{F}' := \{F \in \mathcal{F} : |F| < \infty \text{ and } \sigma(F) \text{ is strongly 1-anchored}\}$$

is cofinal in \mathcal{F} (in the sense that each $F \in \mathcal{F}$ is contained in some $F' \in \mathcal{F}'$), and hence the natural map

$$(19) \quad \operatorname{colim}_{F \in \mathcal{F}'} \pi_0 \overline{F} \longrightarrow \pi_0 M$$

is bijective.

Since $\tilde{\Gamma}$ is a natural transformation, to prove the theorem it will suffice to show that $\tilde{\Gamma}_{\overline{F}}$ is an isomorphism for each $F \in \mathcal{F}'$ and that the natural maps

$$(20) \quad \operatorname{colim}_{F \in \mathcal{F}} \tilde{\Pi}_k(\overline{F}) \longrightarrow \tilde{\Pi}_k(M)$$

and

$$(21) \quad \operatorname{colim}_{F \in \mathcal{F}} \pi_k(\Omega B \overline{F}) \longrightarrow \pi_k(\Omega B M)$$

are isomorphisms for each k .

To show that $\tilde{\Gamma}_{\overline{F}}$ is an isomorphism for each $F \in \mathcal{F}'$, it suffices (by Proposition 2.17) to check that \overline{F} is stably group-like with respect to the element $\sigma(F)$. Letting $F = \{a_1, \dots, a_l\}$, each component of \overline{F} has the form

$$C = [a_1^{n_1} \bullet a_2^{n_2} \cdots \bullet a_l^{n_l}]$$

for some $n_j \geq 0$ ($j = 1, 2, \dots, l$). Setting $N = \max\{n_1, \dots, n_l\}$, we have

$$C \bullet [a_1^{N-n_1} \bullet \cdots \bullet a_l^{N-n_l}] = [\sigma(F)^N],$$

so \overline{F} is stably group-like with respect to $\sigma(F)$.

Next we show that (20) and (21) are isomorphisms. For the former, it suffices to observe that since (19) is a bijection, every map from a path connected space (e.g. S^k or $S^k \times I$) into M factors through one of the embeddings $\overline{F} \hookrightarrow M$. For the latter, it suffices to show that for each k , these embeddings induce an isomorphism

$$(22) \quad \operatorname{colim}_{F \in \mathcal{F}'} \pi_k(B \overline{F}) \xrightarrow{\cong} \pi_k(BM).$$

Given a topological monoid A , the singular simplicial set $S.A$ has the structure of a simplicial object in the category of (discrete) monoids, and we define $B(S.A)$ to be the geometric realization of the bisimplicial set $B.(S.A)$ formed by applying the bar construction to each monoid $S_p A$ ($p \in \mathbb{N}$); thus the set of simplices of $B.(S.A)$ in bi-degree (p, q) is $(S_p A)^q \cong S_p(A^q)$. Let $B(S.A)$ be the simplicial space

$$q \mapsto |S.(A^q)|,$$

so that $|B.(S.A)| = B(S.A)$. One sees that the natural weak equivalences $|S.(A^q)| \xrightarrow{\simeq} A^q$ induce a map $B.(S.A) \rightarrow B.A$, natural in A , and this level-wise weak equivalence induces a weak equivalence

$$B(S.A) \xrightarrow{\epsilon_A} BA$$

on realizations whenever $B.A$ is a proper simplicial space (see Remark 2.3).

Now consider the commutative diagram

$$(23) \quad \begin{array}{ccc} \operatorname{colim}_{F \in \mathcal{F}'} \pi_k B(S.\overline{F}) & \longrightarrow & \pi_k B(S.M) \\ \downarrow \operatorname{colim} (\epsilon_{\overline{F}})_* & & \downarrow (\epsilon_M)_* \\ \operatorname{colim}_{F \in \mathcal{F}'} \pi_k (B\overline{F}) & \longrightarrow & \pi_k (BM), \end{array}$$

Our assumptions on M imply that BM is a proper simplicial space (Remark 2.3), as is each $B\overline{F}$. So the vertical maps in (23) are both isomorphisms.

The bottom map in Diagram (23) is the same as (22), and to prove that this map is an isomorphism, it remains to observe that top map in Diagram (23) is an isomorphism. By (19), each singular simplex $\Delta^n \rightarrow M$ factors through one of the embeddings $\overline{F} \rightarrow M$, and hence the natural map

$$\operatorname{colim}_{F \in \mathcal{F}'} S.\overline{F} \longrightarrow S.M$$

is an isomorphism, as is the induced map

$$\operatorname{colim}_{F \in \mathcal{F}'} B..(S.\overline{F}) \longrightarrow B..(S.M).$$

Since homotopy groups of the geometric realization commute with filtered colimits in the category of (bi)simplicial sets (see [11, Proposition A.2.5.3], for instance), this completes the proof. \square

3. PERMUTATIVE CATEGORIES ARISING FROM GROUP ACTIONS

In this section we introduce a framework for producing permutative categories (internal to **CGTop**) from certain sequences of group actions. This will be used in subsequent sections to give compatible descriptions of deformation K -theory and topological K -theory, facilitating the construction of the topological Atiyah–Segal map as a morphism of spectra. The proofs of the claims made in this section are all routine (and, in fact, relatively short) and will mostly be left to the reader. In Section 3.3, we briefly explain how this theory can be enhanced to produce *bipermutative categories* and hence ring spectra.

We will use the following terminology regarding (topological) group actions: if X is a G -space, Y is an H -space, and $\phi: G \rightarrow H$ is a (continuous) homomorphism, then a map $f: X \rightarrow Y$ is called ϕ -equivariant, or equivariant with respect to ϕ , if

$$f(g \cdot x) = \phi(g) \cdot f(x)$$

for all $g \in G$, $x \in X$.

3.1. Action sequences. The canonical example to keep in mind when reading the following definition is the unitary (or general linear) groups acting on themselves by conjugation, with the usual matrix block sum operations (see Example 3.3).

Definition 3.1. A permutative (left) action sequence is an octuple

$$\mathcal{A} = (I, \{G_i\}_{i \in I}, \{X_i\}_{i \in I}, *, \oplus, \{C_{i,j}\}_{i,j \in I}),$$

where:

- I is a commutative monoid, with identity element 0 and monoid operation $+$;
- $* = *_0 \in X_0$ is a non-degenerate basepoint;
- Each G_i is a topological group with identity element $e_i \in G_i$, and each X_i is a left G_i -space;
- $\oplus = (\oplus^{\text{alg}}, \oplus^{\text{top}})$;
- \oplus^{alg} is an associative collection of homomorphisms

$$\oplus_{i,j}^{\text{alg}}: G_i \times G_j \rightarrow G_{i+j}, \quad i, j \in I;$$

- \oplus^{top} is an associative collection of $\oplus_{i,j}^{\text{alg}}$ -equivariant maps

$$\oplus_{i,j}^{\text{top}}: X_i \times X_j \rightarrow X_{i+j}, \quad i, j \in I,$$

where equivariance refers to the product action of $G_i \times G_j$ on $X_i \times X_j$ (and the action of G_{i+j} on X_{i+j});

- For each $i, j \in I$, we have $C_{i,j} \in G_{i+j}$.

We will usually simplify notation by writing \oplus in place of $\oplus_{i,j}^{\text{alg}}$ or $\oplus_{i,j}^{\text{top}}$.

The elements $C_{i,j}$ are subject to the following further axioms for all $i, j, k \in I$.

- $C_{i,j} \cdot (x_i \oplus x_j) = x_j \oplus x_i$ for each $x_i \in X_i$, $x_j \in X_j$
- $C_{i,0} = C_{0,i} = e_i$;
- $C_{i,j} C_{j,i} = e_{i+j}$;
- If $g_i \in G_i$ and $g_j \in G_j$, then $C_{i,j}(g_i \oplus g_j) = (g_j \oplus g_i)C_{i,j}$;
- $(C_{i,k} \oplus e_j)(e_i \oplus C_{j,k}) = C_{i+j,k}$.

Note that it is not necessary to assume that the basepoint $*_0$ is fixed by the action of G_0 .

We refer to the operations \oplus as the *monoidal*, or *additive*, structure of \mathcal{A} , and we refer to the elements $C_{i,j}$ as the *commutativity operators*.

It will be convenient to use the notation $X = \coprod_i X_i$, $G = \coprod_i G_i$, as well as to set $C = \{C_{i,j}\}$. Then we can write an action sequence in the simplified notation $\mathcal{A} = (I, G, X, *, \oplus, C)$.

A morphism of permutative action sequences

$$\mathcal{A} = (I, G, X, *, \oplus, C) \longrightarrow \mathcal{B} = (J, H, Y, *, \oplus, D)$$

consists of a homomorphism of monoids $f: I \rightarrow J$ together with group homomorphisms

$$\phi_i: G_i \rightarrow H_{f(i)}$$

satisfying $\phi_{i+j}(C_{i,j}) = D_{f(i),f(j)}$, and ϕ_i -equivariant maps

$$\zeta_i: X_i \rightarrow Y_{f(i)}$$

for each $i \in I$, such that $\zeta_0(*) = *$. This defines the category **PAct** of permutative (left) action sequences.

We will always work with left actions, and we drop the adjective *left* from here on. In what follows, we will assume familiarity with the notion of permutative categories (as defined in May [29]).

Construction 3.2. There is a functor

$$\mathcal{T}: \mathbf{PAct} \longrightarrow \mathbf{PCat}$$

from the category of permutative action sequences to the category **PCat** of permutative categories internal to **CGTop**, defined as follows.

Given an action sequence $\mathcal{A} = (I, G, X, *, \oplus, C)$, the object space of $\mathcal{T}(\mathcal{A})$ is simply

$$X = \coprod_{i \in I} X_i,$$

while the morphism space is

$$\coprod_{i \in I} G_i \times X_i.$$

The domain of (g, x) is x , the codomain is $g \cdot x$, and composition is given by

$$(h, g \cdot x) \circ (g, x) = (hg, x).$$

The operations \oplus give rise to a continuous functor

$$\oplus: \mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A}) \longrightarrow \mathcal{T}(\mathcal{A}),$$

which is (strictly) associative and has the object $*_0 \in X_0$ as (strict) unit. The commutativity isomorphisms are given by the morphisms $(C_{i,j}, x_i \oplus x_j)$, and our axioms on the $C_{i,j}$ are exactly what is needed to make the coherence diagrams in $\mathcal{T}(\mathcal{A})$ commute.

We refer to $\mathcal{T}(\mathcal{A})$ as the *translation category* of \mathcal{A} . Note that $\mathcal{T}(\mathcal{A})$ is in fact a groupoid.

Example 3.3. The tautological (additive) unitary permutative action sequence is given by setting $I = \mathbb{N}$, with ordinary addition as the monoid operation, and setting $X_n = G_n = \mathrm{U}(n)$ for $n \in \mathbb{N}$. We define $\mathrm{U}(0) = \{0\}$, the trivial group. Here we view $\mathrm{U}(n)$ as a left $\mathrm{U}(n)$ -space via *conjugation*, and we use the usual matrix block sum operation to define both \oplus^{alg} and \oplus^{top} , with $0 \in \mathrm{U}(0)$ acting as the unit element.

The commutativity operators are given by the (unitary) permutation matrices

$$(24) \quad I_{m,n} = \begin{bmatrix} 0_{nm} & I_n \\ I_m & 0_{mn} \end{bmatrix},$$

where 0_{pq} denotes the $p \times q$ zero matrix.

The tautological additive general linear action sequence is defined similarly, by replacing $U(n)$ by $GL(n)$.

Definition 3.4. An (additive) *unitary permutative action sequence* is one in which the underlying monoid is \mathbb{N} , with its usual addition, and we have $G_n = U(n)$ and $C_{m,n} = I_{m,n}$ for all $m, n \in \mathbb{N}$. Note that such sequences are completely determined by their topological data, that is, the $U(n)$ -spaces X_n (and the basepoint $x_0 \in X_0$) together with the maps \oplus^{top} .

Remark 3.5. The notion of a permutative action sequence can be generalized by allowing the elements $C_{i,j}$ to depend on $x_i \in X_i$ and $x_j \in X_j$ rather than just on $i, j \in I$, and a small modification again gives a functor from this larger category of sequences to **PCat**. Furthermore, there is no need to assume the G_i are groups; monoids would suffice.

3.2. The nerve of a permutative action sequence. Consider a permutative action sequence $\mathcal{A} = (I, G, X, *, \oplus, C)$. The continuous functor

$$\oplus: \mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A}) \longrightarrow \mathcal{T}(\mathcal{A})$$

makes $|N\mathcal{T}(\mathcal{A})|$ into a topological monoid (note here that geometric realization commutes with products of simplicial spaces [35]). May's infinite loop space machine [29] gives a functor **K** from **PCat** to the category of connective Ω -spectra. One key feature of this functor is that for each permutative category \mathcal{C} , the infinite loop space underlying the spectrum **K**(\mathcal{C}) is naturally weakly equivalent to $\Omega B|N\mathcal{C}|$. Our next goal is to give an explicit description of the monoid $|N\mathcal{T}(\mathcal{A})|$.

For a space X with a left action of a topological group G , the homotopy orbit space (also known as the Borel construction) is the quotient space

$$X_{hG} = (EG \times X)/G,$$

where EG is the geometric realization of the category \overline{G} , internal to **CGTop**, with object space G and morphism space $G \times G$ (here (g, h) is the unique morphism from h to g , and $(g, h) \circ (h, k) = (g, k)$). Note that \overline{G} admits a right action of G (by functors), defined via right-multiplication in G . This induces a right action of G on EG , and now G acts on $EG \times X$ via $g \cdot (e, x) = (e \cdot g^{-1}, x)$. When G is a Lie group, the natural map $EG \rightarrow BG$ (induced by the functor sending a morphism (g, h) in \overline{G} to the morphism gh^{-1}) is a universal principal G -bundle [46], and the natural map $X_{hG} \rightarrow BG$ is a fiber bundle with fiber X .

Given a permutative action sequence $\mathcal{A} = (I, G, X, *, \oplus, C)$, we can form

$$\mathcal{M}(\mathcal{A}) := \coprod_{i \in I} (X_i)_{hG_i}.$$

The maps $\oplus_{i,j}^{\text{alg}}$ induce continuous functors

$$\overline{G}_i \times \overline{G}_j \longrightarrow \overline{G}_{i+j}$$

and hence continuous maps

$$(25) \quad EG_i \times EG_j \longrightarrow EG_{i+j}.$$

Since the maps $\oplus_{i,j}^{\text{alg}}$ are homomorphisms, the maps (25) are equivariant (with respect to $\oplus: G_i \times G_j \rightarrow G_{i+j}$). These maps, together with the equivariant maps $\oplus: X_i \times X_j \rightarrow X_{i+j}$, induce a map

$$\mathcal{M}(\mathcal{A}) \times \mathcal{M}(\mathcal{A}) \longrightarrow \mathcal{M}(\mathcal{A}).$$

It is an exercise to check that this map makes $\mathcal{M}(\mathcal{A})$ into a topological monoid with $[\ast, \ast_0] \in (X_0)_{hG_0} \subset \mathcal{M}(\mathcal{A})$ as unit element, where $\ast \in EG$ corresponds to the object in \overline{G} represented by the identity of G .

Proposition 3.6. *There is a natural homeomorphism of topological monoids*

$$\mathcal{M}(\mathcal{A}) \longrightarrow |N.\mathcal{T}(\mathcal{A})|.$$

Proof. (Sketch) A special case of this statement is proven in Ramras [40, Proposition 2.4] (and the argument given there is due to Tyler Lawson). That argument immediately generalizes to produce the desired map and to show that it is a continuous bijection. The argument proceeds by viewing each side as the geometric realization of a simplicial space, and providing a map of simplicial spaces that is a homeomorphism on each level. In [40], an appeal to compactness was made to deduce continuity of the inverse maps on each level, but it is in fact a simple matter to write down explicit formulas for these inverse maps, from which it is clear that the inverses are continuous. \square

Remark 3.7. The commutativity operators $C_{i,j}$ induce a natural transformation between the functors $\oplus: \mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A}) \rightarrow \mathcal{T}(\mathcal{A})$ and $\oplus \circ \tau$, where τ is the twist functor on the product category. It then follows from basic categorical homotopy theory (Segal [46, Proposition 2.1]) that $|N.(\mathcal{T}(\mathcal{A}))| \cong \mathcal{M}(\mathcal{A})$ is homotopy commutative.

Proposition 3.8. *If \mathcal{A} is a unitary or general linear permutative action sequence, then every element in the monoid $\mathcal{M}(\mathcal{A})$ is (strongly) anchored. Consequently, the natural map*

$$\tilde{\Gamma}: \tilde{\Pi}_k \mathcal{M}(\mathcal{A}) \longrightarrow \Omega B\mathcal{M}(\mathcal{A})$$

is an isomorphism for each $k \geq 0$.

Proof. (Sketch) Theorem 2.8 implies that the second statement follows from the first. A special case of the first statement is proven in Ramras [40, Proof of Corollary 4.4], and that argument immediately generalizes. \square

Remark 3.9. The homotopy provided by Remark 3.7 does not anchor elements. The argument in [40, Proof of Corollary 4.4] involves constructing different homotopies, specific to each element we wish to anchor. The main point in the proof is that for each $x \in X_n$, the matrix $I_{n,n}$ lies in the identity component of the stabilizer of x^2 . The fact that this requires no extra assumptions on the stabilizer appears to be a rather special feature of the unitary and general linear groups.

3.3. Bipermutative action sequences. The notion of permutative action sequence introduced here can be extended to a notion of *bipermutative action sequence*, in such a way that the translation category inherits the structure of a bipermutative category. In short, a bipermutative action sequence is a pair of permutative action sequences, sharing the same indexing set I , the same spaces X_i , and the same groups G_i . Additional coherence axioms relating the two permutative structures must hold (and these axioms imply that the two monoid structures on I give it the structure of a rig, or a “ring without negatives”). We give the details in Definition 3.10 below; the axioms are just direct translations of the axioms for bipermutative categories. Maps of bipermutative action sequences are just maps that respect both permutative structures.

As an example, the Kronecker product of matrices endows the tautological unitary and general linear action sequences with a bipermutative structure. The details are just an elaboration of the discussion in May [33, VI §5]. We note that some care must be taken when specifying the exact definition of Kronecker product, so that the coherence axioms hold.

Definition 3.10. A bipermutative action sequence is rig R together with a pair of action sequences

$$((R, +), G, X, *_0, \oplus, C)$$

and

$$((R, \cdot), G, X, *_1, \otimes, D)$$

sharing the same groups G_r and the same G_r -spaces X_r for all $r \in R$. Let $0 \in R$ and $1 \in R$ denote the additive and multiplicative identity elements of R , respectively.

These data must satisfy the following additional axioms for all $r, s, t, u \in R$ and all $x, y, z, w \in X$ and all $g, h, k, l \in G$:

- **Zero Axioms:** $*_0 \otimes x = *_0 = x \otimes *_0$ and $e_0 \otimes g = e_0 = g \otimes e_0$ (recall that $e_0 \in G_0$ is the identity element);
- **Right Distributivity Axioms:** $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ and $(g \oplus h) \otimes k = (g \otimes k) \oplus (h \otimes k)$;

- Coherence Axioms: $C_{r,s} \otimes e_t = C_{r \cdot t, s \cdot t}$, and

$$(D_{t,r+s} \oplus D_{u,r+s})D_{r+s,t+u} = (e_{r \cdot t} \oplus C_{r \cdot u, s \cdot t} \oplus e_{s \cdot u}) [[(D_{t,r} \oplus D_{u,r})D_{r,t+u}] \oplus [(D_{t,s} \oplus D_{u,s})D_{s,t+u}]].$$

We will sometimes denote these sequences in the simplified form

$$(R, G, X, *_0, *_1, \oplus, C, \otimes, D).$$

A morphism of bipermutative action sequences

$$(R, G, X, *_0, *_1, \oplus, C, \otimes, D) \longrightarrow (S, H, Y, *_0, *_1, \oplus, C', \otimes, D')$$

consists of a function $f: R \rightarrow S$ that is a monoid homomorphism for both $+$ and \cdot , together with homomorphisms $\phi_r: G_r \rightarrow H_{f(r)}$ and ϕ_r -equivariant maps $X_r \rightarrow Y_{f(r)}$ (preserving both basepoints) for all $r \in R$. The homomorphisms ϕ_r must satisfy $\phi_{s+t}(C_{s,t}) = C'_{f(s+t)}$ and $\phi_{s \cdot t}(D_{s,t}) = D'_{f(s \cdot t)}$.

Bipermutative action sequence now form a category **BPAct**, and the translation category construction provides a functor from **BPAct** to the category of bipermutative categories (internal to **CGTop**).

Multiplicative infinite loop space theory, as developed by May [33, 31, 32], provides a functor taking a bipermutative category \mathcal{C} to a (connective) E_∞ ring spectrum $\mathbb{K}_\infty(\mathcal{C})$. Just as in the permutative case, the underlying infinite loop space $\Omega^\infty \mathbb{K}_\infty(\mathcal{C})$ is naturally weakly equivalent to the group completion ΩBC , where BC denotes the geometric realization of the bar construction applied to the nerve of \mathcal{C} , using its *additive* monoidal structure (the key step in the proof of this statement is [32, Theorem 9.3]).

4. DEFORMATION K -THEORY

We can use Theorem 2.8 to describe the homotopy groups of the unitary and general linear deformation K -theory spectra associated to finitely generated discrete groups. We explain how to view the construction of these objects from [40] in terms of action sequences. We will focus on the unitary version; the definitions and statements in general linear version are closely analogous, and we will explain the necessary modifications to the proofs that are required in the general linear case.

Definition 4.1. Given a discrete group G , let $\mathcal{A}(G)$ denote the unitary permutative action sequence associated to the spaces $X_n = \text{Hom}(G, \text{U}(n))$, which we topologize as subsets of the mapping spaces $\text{Map}(G, \text{U}(n))$. Note that $\text{Hom}(G, \text{U}(0))$ consists of a single point, which will serve as $*$. We let the unitary groups act on these spaces by conjugation, and the block sum operations are induced by block sum of matrices. Note that $\mathcal{A}(G)$ is contravariantly functorial in G .

The deformation K -theory spectrum of G is the connective Ω -spectrum $K^{\text{def}}(G)$ associated to the permutative translation category $\mathcal{T}(\mathcal{A}(G))$, and

we define

$$K_*^{\text{def}}(G) = \pi_* K^{\text{def}}(G) \cong \pi_* \Omega^\infty K^{\text{def}}(G).$$

By Proposition 3.6, we have a natural homeomorphism

$$\Omega^\infty K^{\text{def}}(G) \cong \Omega B \left(\coprod_{n=0}^{\infty} \text{Hom}(G, \text{U}(n))_{h\text{U}(n)} \right),$$

where the coproduct on the right has the topological monoid structure induced from the additive structure of $\mathcal{A}(G)$. To simplify notation, we define

$$\text{Rep}(G)_{h\text{U}} = \coprod_{n=0}^{\infty} \text{Hom}(G, \text{U}(n))_{h\text{U}(n)}.$$

Proposition 3.8 implies that for each $m \geq 0$, the natural map

$$\tilde{\Gamma}: \tilde{\Pi}_m(\text{Rep}(G)_{h\text{U}}) \longrightarrow K_m^{\text{def}}(G)$$

is an isomorphism. Our next goal is to describe these homotopy groups explicitly in terms of spherical families of representations $S^m \rightarrow \text{Hom}(G, \text{U}(n))$.

Definition 4.2. Given a discrete group G , define

$$\text{Rep}(G) = \text{Rep}(G, \text{U}) := \coprod_{n=0}^{\infty} \text{Hom}(G, \text{U}(n)).$$

Block sum of matrices makes $\text{Rep}(G)$ into a topological monoid, with the unique element in $\text{Hom}(G, \text{U}(0))$ as the identity. Replacing $\text{U}(n)$ by $\text{GL}(n)$, we obtain the monoid $\text{Rep}(G, \text{GL})$.

Lemma 4.3. *Let G be a discrete group. Then each component in $\text{Rep}(G, \text{U})$ is (strongly) anchored, and the same holds for $\text{Rep}(G, \text{GL})$.*

Proof. We will phrase the proof so as to apply to both the unitary and general linear cases. Consider an n -dimensional representation ρ . Since the matrix $I_{n,n}$ defined in (24) is diagonalizable, Ramras [40, Lemma 4.3] implies that there exists a path A_t in $\text{Stab}(\rho \oplus \rho)$ (the stabilizer under the conjugation action) with $A_0 = I_{2n}$ and $A_1 = I_{n,n}$. Now

$$(\psi, \psi') \mapsto A_t(\psi \oplus \psi')A_t^{-1}$$

defines a homotopy anchoring ρ . □

Let $K^{-m}(\ast)$ denote the complex K -theory of a point, so $K^{-m}(\ast) = \mathbb{Z}$ for m even and $K^{-m}(\ast) = 0$ for m odd.

Theorem 4.4. *Let G be a discrete group. Then there are natural isomorphisms*

$$K_m^{\text{def}}(G) \cong \tilde{\Pi}_m(\text{Rep}(G)) \oplus K^{-m}(\ast)$$

for each $m > 0$, as well natural isomorphisms

$$K_0^{\text{def}}(G) \cong \tilde{\Pi}_0(\text{Rep}(G)) \cong \text{Gr}(\pi_0 \text{Rep}(G)).$$

Analogous isomorphisms exist in the general linear case, so long as G is finitely generated.

We will need a lemma regarding the fibrations

$$(26) \quad \mathrm{Hom}(G, \mathrm{U}(n)) \xrightarrow{i_n} \mathrm{Hom}(G, \mathrm{U}(n))_{h\mathrm{U}(n)} \xrightarrow{q_n} \mathrm{BU}(n).$$

Lemma 4.5. *Let $\rho: G \rightarrow \mathrm{U}(n)$ be a representation of the form $\rho \cong \rho' \oplus I_k$, and assume that ρ' decomposes further as a direct sum of the form*

$$\rho' = \bigoplus_i k\rho'_i$$

for some representations ρ'_i , where $k\rho'_i$ denotes the k -fold block sum of ρ'_i with itself. Then for $m \leq 2k$, the map i_n appearing in the fibration sequence (26) induces an injection

$$(i_n)_*: \pi_m(\mathrm{Hom}(G, \mathrm{U}(n)), \rho) \hookrightarrow \pi_m(\mathrm{Hom}(G, \mathrm{U}(n))_{h\mathrm{U}(n)}, i_n(\rho)).$$

The analogous statement holds with $\mathrm{U}(n)$ replaced by $\mathrm{GL}(n)$, so long as G is finitely generated.

Proof. We begin by proving the unitary case. At the end, we will explain the additional arguments needed in the general linear case.

If $k = n$ (that is, if $\rho = I_n$), then this follows from the fact that the fibration (26) admits a splitting sending $[e, *] \in \mathrm{BU}(n) \cong \{*\}_{h\mathrm{U}(n)}$ to $[e, I_n] \in \mathrm{Hom}(G, \mathrm{U}(n))_{h\mathrm{U}(n)}$; this splitting is just the map on homotopy orbit spaces induced by the $\mathrm{U}(n)$ -equivariant inclusion $\{I_n\} \hookrightarrow \mathrm{Hom}(G, \mathrm{U}(n))$ (note that in this case, $(i_n)_*$ is injective in all dimensions, not just when $m \leq 2k = 2n$). So we will assume $k < n$.

It will suffice to show that for $m \leq 2k$, the boundary map

$$\partial: \pi_{m+1}(\mathrm{BU}(n), *) \longrightarrow \pi_m(\mathrm{Hom}(G, \mathrm{U}(n)), \rho)$$

is zero. Since we have assumed $k < n$, we have $m \leq 2k < 2n$, which means $\pi_{m+1}(\mathrm{BU}(n), *)$ is in the stable range, and is zero for m even. Hence we will assume m is odd for the remainder of the proof.

Let $O_\rho \subset \mathrm{Hom}(G, \mathrm{U}(n))$ denote the conjugation orbit of ρ . Then ∂ factors through the boundary map for the fibration

$$O_\rho \longrightarrow (O_\rho)_{h\mathrm{U}(n)} \longrightarrow \mathrm{BU}(n),$$

and we claim that $\pi_m(O_\rho) = 0$ if m is odd and less than $2k$, which will complete the proof. Letting $\mathrm{Stab}_\rho \leq \mathrm{U}(n)$ denote the stabilizer of ρ under conjugation, Schur's Lemma implies that

$$(27) \quad \mathrm{Stab}_\rho \cong \prod_i \mathrm{U}(n_i),$$

where the numbers n_i are the multiplicities of the irreducible summands of ρ . Our assumption on ρ implies that $n_i \geq k$ for each i .

The inclusion

$$\mathrm{Stab}(I_k) \cong \mathrm{U}_k \hookrightarrow \mathrm{Stab}_\rho \hookrightarrow \mathrm{U}(n)$$

is homotopic to the standard inclusion of $\mathrm{U}(k)$ into $\mathrm{U}(n)$, and hence is an isomorphism on homotopy in dimensions less than $2k$. This implies that the

inclusion $\text{Stab}_\rho \hookrightarrow \text{U}(n)$ is surjective on homotopy in dimensions less than $2k$, and consequently the boundary map for the fibration sequence

$$\text{Stab}_\rho \longrightarrow \text{U}(n) \xrightarrow{q} O_\rho$$

is an injection

$$\pi_m(O_\rho) \hookrightarrow \pi_{m-1}(\text{Stab}_\rho)$$

for $m \leq 2k - 1$. But we are assuming that m is odd and $m < 2k \leq 2n_i$ for each i , so

$$\pi_{m-1}\text{Stab}_\rho \cong \prod_i \pi_{m-1}\text{U}(n_i) = 0.$$

It follows that $\pi_m(O_\rho) = 0$ when m is odd and less than $2k$. This completes the proof in the unitary case.

To extend this argument to the general linear case, we need to analyze Stab_ρ for representations $\rho: G \rightarrow \text{GL}(n)$, with G finitely generated. If ρ is completely reducible (that is, isomorphic to a block sum of irreducible representations) then a decomposition analogous to (27) still holds, but this can fail if ρ is not completely reducible. We claim there exists a representation in the path component of ρ which is completely reducible and still satisfies the hypotheses of the Lemma. This will suffice, since the map $(i_n)_*$ is independent (up to isomorphism) of the chosen basepoint within the path component of ρ . Note that the rest of the unitary argument applies equally well in the general linear case, since $\text{GL}(n) \simeq \text{U}(n)$.

To prove this statement, we will appeal to some results from algebraic geometry. Since G is finitely generated, $\text{Hom}(G, \text{GL}(n))$ is an affine variety. To see this, first, note that

$$\text{GL}(n) \cong \{(A, B) \in M_{n \times n} \mathbb{C} \cong \mathbb{C}^{n^2} : AB = I\}$$

is an affine variety. Now if G is generated by l elements, then $\text{Hom}(G, \text{GL}(n))$ is cut out from $\text{GL}(n)^l$ by the ideal of polynomials corresponding to the relations in G .

A basic result in Geometric Invariant Theory states for every conjugation orbit $O \subset \text{Hom}(G, \text{GL}(n))$, there exists a (unique) completely reducible representation inside the closure \overline{O} (in general, orbit closures of complex reductive groups acting on affine varieties contain unique closed orbits [10, Corollary 6.1 and Theorem 6.1], and in the present situation complete reducibility is equivalent to having a closed orbit [27, Theorem 1.27]). This implies that every path component of $\text{Hom}(G, \text{GL}(n))$ contains a completely reducible representation, since $\text{Hom}(G, \text{GL}(n))$ is triangulable (as are all affine varieties [17]).

Recall that we have a decomposition

$$\rho = \left(\bigoplus_i k\rho'_i \right) \oplus I_k.$$

Letting ρ_i'' denote the completely reducible representation in the path component of ρ_i' , we see that

$$\left(\bigoplus_i k\rho_i'' \right) \oplus I_k$$

is again completely reducible, lies in the same path component as ρ , and satisfies the hypotheses of the Lemma, as desired. \square

Proof of Theorem 4.4. For $m = 0$, this is elementary (see the proof of [40, Lemma 2.5]), so we will assume $m > 0$. We will work in the unitary case; the proof in the general linear case is the same.

Lemma 4.3 and Theorem 2.8 give a natural isomorphism

$$\pi_m \Omega B \text{Rep}(G) \cong \tilde{\Pi}_m(\text{Rep}(G)).$$

To obtain natural isomorphisms between

$$K_m^{\text{def}}(G) = \pi_m \Omega B(\text{Rep}(G)_{hU})$$

and

$$\tilde{\Pi}_m(\text{Rep}(G)) \oplus K^{-m}(*),$$

($m > 0$) it remains only to show that there are natural isomorphisms

$$(28) \quad \pi_m \Omega B(\text{Rep}(G)_{hU}) \cong \pi_m \Omega B(\text{Rep}(G)) \oplus K^{-m}(*).$$

Each representation space for the trivial group is a single point, so

$$\text{Rep}(\{1\})_{hU} \cong \coprod_n BU(n),$$

and this monoid is stably group-like with respect to each of its components. Hence the Group Completion Theorem, together with Bott Periodicity, gives us isomorphisms

$$\pi_m \Omega B(\text{Rep}(\{1\})_{hU}) \cong K^{-m}(*),$$

and this will be our model for $K^{-m}(*)$.

The inclusion $\{1\} \hookrightarrow G$ induces a map of monoids

$$(29) \quad q: \text{Rep}(G)_{hU} \longrightarrow \text{Rep}(\{1\})_{hU},$$

and hence a map

$$\Omega B q: \Omega B(\text{Rep}(G)_{hU}) \longrightarrow \Omega B(\text{Rep}(\{1\})_{hU}).$$

The maps

$$(30) \quad i_n: \text{Hom}(G, U(n)) \hookrightarrow \text{Hom}(G, U(n))_{hU(n)}$$

combine into a monoid homomorphism

$$i: \text{Rep}(G) \hookrightarrow \text{Rep}(G)_{hU},$$

and we have an induced map

$$\Omega B i: \Omega B \text{Rep}(G) \longrightarrow \Omega B(\text{Rep}(G)_{hU}).$$

To complete the proof, it will suffice to show that the sequence
(31)

$$0 \longrightarrow \pi_m \Omega B \text{Rep}(G) \xrightarrow{i_*} \pi_m \Omega B \text{Rep}(G)_{hU} \xrightarrow{q_*} \pi_m \Omega B \text{Rep}(\{1\})_{hU} \longrightarrow 0$$

is split exact for each $m \geq 1$ (note that we are abbreviating $(\Omega Bi)_*$ to i_* and $(\Omega Bq)_*$ to q_*). The map q_* admits a right inverse, induced by the projection $G \rightarrow \{1\}$, so the sequence (31) splits and q_* is surjective. Next, the composite $q \circ i$ factors through the discrete monoid \mathbb{N} . Since $\pi_m \Omega B \mathbb{N} = 0$ for $m > 0$, we have $q_* \circ i_* = 0$. To prove exactness of (31), it remains to show that $\ker(q_*) \subset \text{Im}(i_*)$ and $\ker(i_*) = 0$. We will prove these statements by direct argument using Theorem 2.8, which provides an isomorphism between (31) and the corresponding sequence obtained by applying the functor $\tilde{\Pi}_m$ in place of $\pi_m \Omega B$. Hence for the rest of the argument we will work with the sequence

$$(32) \quad 0 \longrightarrow \tilde{\Pi}_m \text{Rep}(G) \xrightarrow{i_*} \tilde{\Pi}_m (\text{Rep}(G)_{hU}) \xrightarrow{q_*} \tilde{\Pi}_m (\text{Rep}(\{1\})_{hU}) \longrightarrow 0.$$

First we show that $\ker(q_*) \subset \text{Im}(i_*)$. By Proposition 2.10, each element in $\ker(q_*)$ is represented by a map $\psi: S^m \rightarrow \text{Hom}(G, U(k))_{hU(k)}$ such that for some constant map $c: S^m \rightarrow BU(k')$, the map $(q \circ \psi) \oplus c$ is nullhomotopic. It follows that for any constant map $\tilde{c}: S^m \rightarrow \text{Hom}(G, U(k'))_{hU(k')}$, the homotopy class $\langle \psi \oplus \tilde{c} \rangle$ lies in the kernel of

$$q_*: \pi_* (\text{Hom}(G, U(k+k'))_{hU(k+k')}) \longrightarrow \pi_* BU(k+k')$$

(for appropriately chosen basepoints). Since the sequence (26) is a fibration sequence for each $n \geq 0$, there exists a map

$$\rho: S^m \rightarrow \text{Hom}(G, U(k+k'))$$

such that $i_n \circ \rho \simeq \psi \oplus \tilde{c}$, and this shows that $[\rho]$ is in the image of the map i_* in (32).

The proof that $\ker(i_*) = 0$ is similar, but will require Lemma 4.5. Each element in $\ker(i_*)$ is represented by a map $\rho: S^m \rightarrow \text{Hom}(G, U(n))$ such that

$$(33) \quad (i \circ \rho) \oplus d \simeq c$$

for some constant maps $c, d: S^m \rightarrow \text{Rep}(G)_{hU}$. Let $m\rho(1) \oplus I_m$ denote the constant map $S^m \rightarrow \text{Rep}(G)$ with image $m\rho(1) \oplus I_m$. Adding

$$md \oplus mc \oplus i((m-1)\rho(1) \oplus I_m),$$

to both sides of (33) gives

$$(34) \quad \begin{aligned} (i \circ \rho) \oplus (m+1)d \oplus mc \oplus i((m-1)\rho(1) \oplus I_m) \\ \simeq (m+1)c \oplus md \oplus i((m-1)\rho(1) \oplus I_m). \end{aligned}$$

Since $U(n)$ is path connected, the map i_n induces a bijection on path components for every n . Thus there exist constant maps \tilde{c} and \tilde{d} such that

$$(35) \quad i \circ \tilde{c} \simeq c \quad \text{and} \quad i \circ \tilde{d} \simeq d.$$

Setting

$$e = (m+1)\tilde{d} \oplus m\tilde{c} \oplus (m-1)\rho(1) \oplus I_m,$$

Equation (34) gives

$$(36) \quad i \circ (\rho \oplus e) \simeq i \circ ((m+1)\tilde{c} \oplus m\tilde{d} \oplus (m-1)\rho(1) \oplus I_m),$$

so in particular $i \circ (\rho \oplus e)$ is nullhomotopic (in the based sense, in fact). Moreover, we have an isomorphism of representations

$$\rho(1) \oplus e(1) \cong m\rho(1) \oplus (m+1)\tilde{d}(1) \oplus m\tilde{c}(1) \oplus I_m,$$

so Lemma 4.5 implies that the map

$$\pi_m(\mathrm{Hom}(G, \mathrm{U}(n))) \xrightarrow{(i_n)_*} \pi_m(\mathrm{Hom}(G, \mathrm{U}(n))_{h\mathrm{U}(n)})$$

is injective if we use $(\rho \oplus e)(1)$ as our basepoint for $\mathrm{Hom}(G, \mathrm{U}(n))$. Since $i \circ (\rho \oplus e)$ is nullhomotopic, we conclude that $\rho \oplus e$ is nullhomotopic as well, and since e is constant it follows that $[\rho] = 0$ in $\tilde{\Pi}_m \mathrm{Rep}(G)$. This completes the proof that (32) is exact, and also completes the proof of the Corollary. \square

It will also be helpful to consider a reduced form of deformation K -theory. The unitary and general linear cases of this discussion are identical.

For each group G , the map $\{1\} \rightarrow G$ induces a map of spectra

$$q: K^{\mathrm{def}}(G) \rightarrow K^{\mathrm{def}}(\{1\}),$$

which admits a splitting induced by the map $G \rightarrow \{1\}$. The map q is just the spectrum level version of the map (29). Note that $K^{\mathrm{def}}(\{1\})$ is simply the connective K -theory spectrum \mathbf{ku} (see May [33, VIII §2], for instance).

Definition 4.6. We define $\tilde{K}^{\mathrm{def}}(G)$ to be the homotopy fiber of the natural map $K^{\mathrm{def}}(G) \rightarrow K^{\mathrm{def}}(\{1\})$, and we set $\tilde{K}_*^{\mathrm{def}}(G) = \pi_* \tilde{K}^{\mathrm{def}}(G)$.

Corollary 4.7. *There is a natural splitting*

$$K_m^{\mathrm{def}}(G) \cong \tilde{K}_m^{\mathrm{def}}(G) \oplus K^{-m}(*)$$

for each $m \geq 0$, and for $m > 0$ there are natural isomorphisms

$$\tilde{K}_m^{\mathrm{def}}(G) \cong \tilde{\Pi}_m \mathrm{Rep}(G) \cong \pi_m \Omega B \mathrm{Rep}(G).$$

Additionally, $\tilde{K}_0^{\mathrm{def}}(G)$ is naturally isomorphic to the quotient of

$$\mathrm{Gr}(\pi_0 \mathrm{Rep}(G)) \cong \pi_0 \Omega B \mathrm{Rep}(G)$$

by the subgroup generated by the trivial 1-dimensional representation.

Proof. The splitting is immediate from the definitions. Also, our definition of $\tilde{K}^{\mathrm{def}}(G)$ implies that $\tilde{K}_m^{\mathrm{def}}(G)$ is naturally isomorphic to the kernel of the surjection $K_m^{\mathrm{def}}(G) \rightarrow K_m^{\mathrm{def}}(\{1\}) \cong K^{-m}(*)$, and exactness of (32) shows that for each $m > 0$, this kernel is naturally isomorphic to $\tilde{\Pi}_m \mathrm{Rep}(G)$.

When $m = 0$, the splitting gives a natural isomorphism between $\tilde{K}_0^{\mathrm{def}}(G)$ and the cokernel of the map $K_0^{\mathrm{def}}(\{1\}) \rightarrow K_0^{\mathrm{def}}(G)$, whose image is generated by the trivial 1-dimensional representation. \square

5. TOPOLOGICAL K -THEORY

In this section, $X = (X, x_0)$ will denote a (based) path connected, paracompact space having the homotopy type of compact Hausdorff space. (For instance, X might be $B\pi_1(K)$ for some aspherical finite CW complex K .)

We will define a permutative category whose homotopy groups agree with the complex topological K -theory of X . Our construction, and the subsequent discussion, is designed to mirror the construction of deformation K -theory in the previous section. This will facilitate our construction and analysis of the topological Atiyah–Segal map in the next section. As discussed in Section 9.1, the permutative category defined below can actually be given a *bipermutative* structure. While this will be important for the results in Section 9, we note that it is unclear whether the induced ring structure agrees with the classical multiplication in K -theory (induced by tensor product of vector bundles) in general.

As in the previous section, there is both a unitary and a general linear version of the constructions given here. We focus on the unitary case; the general linear case is completely analogous (more so than for deformation K -theory).

Definition 5.1. Let $BU(n)$ denote the geometric realization of the one-object category $\underline{U}(n)$ (as in Section 2). Then the (left) conjugation action of $\underline{U}(n)$ on itself induces an action, by continuous functors, of $\underline{U}(n)$ on the category $\underline{U}(n)$, and hence an action of $\underline{U}(n)$ on $BU(n)$. This in turn induces an action of $\underline{U}(n)$ on the based mapping space $\text{Map}_*(X, BU(n))$. Throughout this section, we will view $\text{Map}_*(X, BU(n))$ as a $\underline{U}(n)$ -space under this action.

The block sum operations $\oplus: \underline{U}(m) \times \underline{U}(n) \rightarrow \underline{U}(m+n)$ are homomorphisms, and hence induce continuous functors $\underline{U}(m) \times \underline{U}(n) \rightarrow \underline{U}(m+n)$, which realize to maps

$$BU(m) \times BU(n) \longrightarrow BU(m+n),$$

and induce equivariant maps

$$\oplus: \text{Map}_*(X, BU(m)) \times \text{Map}_*(X, BU(n)) \longrightarrow \text{Map}_*(X, BU(m+n)).$$

Functoriality of B implies that this data gives a unitary permutative action sequence $\mathcal{A}_K(X)$ with n th space $\text{Map}_*(X, BU(n))$.

Note that when $X = *$, we recover the unitary permutative action sequence whose associated spectrum is \mathbf{ku} .

Let $\mathcal{C}_K(X) = \mathcal{T}(\mathcal{A}_K(X))$ be the translation category of $\mathcal{A}_K(X)$, and let $\mathcal{K}(X)$ denote the associated spectrum. Let $\tilde{\mathcal{K}}(X)$ denote the homotopy fiber of the natural map $\mathcal{K}(X) \rightarrow \mathcal{K}(*)$. Finally, set $\mathcal{K}_*(X) = \pi_*\mathcal{K}(X)$ and $\tilde{\mathcal{K}}_*(X) = \pi_*\tilde{\mathcal{K}}(X)$.

We have the following consequence of Propositions 3.6 and 3.8.

Corollary 5.2. *The geometric realization of the nerve of $\mathcal{C}_K(X)$ is isomorphic, as a topological monoid, to the topological monoid*

$$\mathcal{V}(X)_{hU} := \coprod_n \text{Map}_*(X, BU(n))_{hU(n)},$$

and the natural map

$$\tilde{\Gamma}: \tilde{\Pi}_m(\mathcal{V}(X)_{hU}) \longrightarrow \pi_m \Omega B(\mathcal{V}(X)_{hU}) \cong \mathcal{K}_m(X)$$

is an isomorphism for each $m \geq 0$.

Our goal in this section is to compare the homotopy groups $\mathcal{K}_*(X)$ with the (complex) topological K -theory of X (for $* \geq 0$).

We need to specify a definition of topological K -theory. Note that the Group Completion Theorem gives a natural homotopy equivalence

$$\mathbb{Z} \times BU \longrightarrow \Omega B \left(\coprod_n BU(n) \right),$$

where $BU = \text{colim}_n BU(n)$, the colimit being formed with respect to the maps induced by block sum with the identity $I_1 \in U(1)$.

Definition 5.3. For $m \geq 0$, we define the *reduced* topological K -theory of X by

$$\tilde{K}^{-m}(X) = \tilde{K}^0(S^m \wedge X) = \langle S^m \wedge X, \mathbb{Z} \times BU \rangle$$

and the unreduced K -theory of X is defined by

$$K^{-m}(X) = \tilde{K}^{-m}(X) \oplus K^{-m}(*) = \tilde{K}^{-m}(X) \oplus \pi_m(\mathbf{ku}).$$

Maps $X \rightarrow Y$ induce maps on both reduced and unreduced K -theory (in the latter case, all maps act as the identity on $K^{-m}(*)$).

We will need to consider another topological monoid related to $\mathcal{K}(X)$.

Definition 5.4. Define

$$\mathcal{V}(X) = \coprod_n \text{Map}_*(X, BU(n)),$$

and equip $\mathcal{V}(X)$ with the monoid structure induced by the block sum operations on $\{BU(n)\}_n$ described in Definition 5.1.

We now have the following analogue of the results from Section 4.

Corollary 5.5. *For each $m \geq 0$, there is a natural splitting*

$$\mathcal{K}_m(X) \cong \tilde{\mathcal{K}}_m(X) \oplus \tilde{\mathcal{K}}_m(*) = \tilde{\mathcal{K}}_m(X) \oplus \pi_m \mathbf{ku}$$

and a natural isomorphism

$$\mathcal{K}_m(X) \cong K^{-m}(X),$$

which restricts to an isomorphism

$$\tilde{\mathcal{K}}_m(X) \cong \tilde{K}^{-m}(X).$$

Moreover, for $m > 0$ there are natural isomorphisms

$$\tilde{\mathcal{K}}_m(X) \cong \tilde{\Pi}_m \mathcal{V}(X) \cong \pi_m \Omega B\mathcal{V}(X),$$

and $\tilde{\mathcal{K}}_0(X) \cong \tilde{K}^0(X)$ is naturally isomorphic to the quotient of

$$\mathrm{Gr}(\pi_0 \mathcal{V}(X)) \cong \pi_0 \Omega B\mathcal{V}(X)$$

by the subgroup generated by the class of nullhomotopic $X \rightarrow BU(1)$.

The proof is analogous to the arguments in Section 4, but simpler. The technical arguments in the proof of Lemma 4.5 are designed to show that each component of $\mathrm{Rep}(G)$ is a summand of a component on which the fibration (26) is well-behaved. In the present context, each component of $\mathcal{V}(X)$ is a summand of a component of nullhomotopic maps: indeed, since X is homotopy equivalent to a compact Hausdorff space K , each map $f: X \rightarrow BU(n)$ corresponds to a vector bundle over K , which is a direct summand of a trivial bundle, and it follows that there is a map $f': X \rightarrow BU(m)$ (for some m) such that $f \oplus f'$ is nullhomotopic. The desired statements about the fibration

$$\mathrm{Map}_*(X, BU(n)) \longrightarrow (\mathrm{Map}_*(X, BU(n)))_{hU(n)} \longrightarrow BU(n)$$

now follow from the existence of the splitting

$$BU(n) \rightarrow (\mathrm{Map}_*(X, BU(n)))_{hU(n)},$$

which is simply the map on homotopy orbit spaces induced by the equivariant map $\{c_*\} \rightarrow \mathrm{Map}_*(X, BU(n))$. Note here that the constant map c_* to the basepoint $* \in BU(n)$ is fixed under conjugation, since the same is true of $*$ itself.

Remark 5.6. Corollary 5.5 requires the assumption that X is path connected. Indeed, the homotopy group completion of $\mathcal{V}_*(X)_{hU}$ has the form $\mathbb{Z} \times \mathrm{colim}_{n \rightarrow \infty} \mathrm{Map}_*(X, BU(n))_{hU(n)}$ (up to homotopy). For every space X and every $m \geq 1$, we have

$$\begin{aligned} \pi_m \left(\mathbb{Z} \times \mathrm{colim}_{n \rightarrow \infty} \mathrm{Map}_*(X, BU(n))_{hU(n)} \right) \\ &\cong \pi_m \mathrm{Map}_*(X, BU) \oplus \pi_m BU \\ &= \pi_m \mathrm{Map}_*(X, \mathbb{Z} \times BU) \oplus \pi_m BU \\ &= K^{-m}(X), \end{aligned}$$

but when $m = 0$ there is a discrepancy if X is disconnected.

6. THE TOPOLOGICAL ATIYAH–SEGAL MAP

Let G be a group whose classifying space BG has the homotopy type of a finite CW complex. We now define reduced and unreduced versions of the topological Atiyah–Segal map, which relates the deformation K -theory

of G to the topological K –theory of BG . The unitary and general linear discussions are completely parallel, so we focus on the unitary case.

We will see that in dimension zero, the classical Atiyah–Segal map, which associates to each representation $\rho: G \rightarrow \mathrm{U}(n)$ the K –theory class represented by the vector bundle $E_\rho \rightarrow BG$, factors as

$$R[G] = \mathrm{Gr}(\mathrm{Rep}(G)^\delta) \longrightarrow \mathrm{Gr}(\pi_0 \mathrm{Rep}(G)) \xrightarrow{\alpha_0} K^0(BG),$$

where $\mathrm{Rep}(G)^\delta$ is the discrete monoid underlying the topological monoid $\mathrm{Rep}(G)$, and α_0 is the topological Atiyah–Segal map (in dimension 0).

In this section we will discuss additive structures only, postponing the discussion of multiplicative structures to Section 9.

6.1. Additive structure of the topological Atiyah–Segal map. The simplicial classifying space functor B induces continuous, $\mathrm{U}(n)$ –equivariant maps

$$(37) \quad \begin{array}{ccc} B = B_n: \mathrm{Hom}(\Gamma, \mathrm{U}(n)) & \longrightarrow & \mathrm{Map}_*(B\Gamma, BU(n)) \\ \rho & \longmapsto & B\rho \end{array}$$

which combine to give a map between the associated unitary permutative action sequences. Recall that the spectra associated to these action sequences are $K^{\mathrm{def}}(G)$ and $\mathcal{K}(BG)$, respectively, and the homotopy groups of the latter are the complex K –theory groups of BG .

Definition 6.1. The *topological Atiyah–Segal map*

$$\alpha = \alpha^G: K^{\mathrm{def}}(G) \longrightarrow \mathcal{K}(BG)$$

is the map of spectra induced by the above map of permutative action sequences. The reduced topological Atiyah–Segal map

$$\tilde{\alpha}: \tilde{K}^{\mathrm{def}}(G) \longrightarrow \tilde{\mathcal{K}}(BG)$$

is the induced map

$$\mathrm{hofib}\left(K^{\mathrm{def}}(G) \rightarrow K^{\mathrm{def}}(\{1\})\right) \longrightarrow \mathrm{hofib}(\mathcal{K}(BG) \rightarrow \mathcal{K}(*)).$$

Note that we have $K^{\mathrm{def}}(\{1\}) = \mathcal{K}(*) = \mathbf{ku}$.

The results in the previous sections combine to give the following descriptions of the induced maps α_* and $\tilde{\alpha}_*$ on homotopy groups.

Corollary 6.2. *For $m \geq 0$, the topological Atiyah–Segal map is naturally isomorphic to the maps*

$$\pi_m \Omega B\mathrm{Rep}(G)_{h\mathrm{U}} \longrightarrow \pi_m \Omega B\mathcal{V}(BG)_{h\mathrm{U}}$$

and

$$\tilde{\Pi}_m \mathrm{Rep}(G)_{h\mathrm{U}} \longrightarrow \tilde{\Pi}_m \Omega B\mathcal{V}(BG)_{h\mathrm{U}}.$$

induced by the simplicial classifying space functor B , and there is a natural splitting

$$(38) \quad \alpha_* = \tilde{\alpha}_* \oplus \mathrm{Id}_{\pi_* \mathbf{ku}}.$$

Moreover, for $m > 0$, the reduced Atiyah–Segal map is naturally isomorphic to the maps

$$\pi_m \Omega B \operatorname{Rep}(G) \longrightarrow \pi_m \Omega B \mathcal{V}(BG)$$

and

$$\tilde{\Pi}_m \operatorname{Rep}(G) \longrightarrow \tilde{\Pi}_m \mathcal{V}(BG)$$

induced by B .

We can give an explicit description of the topological Atiyah–Segal map in terms of vector bundles. Consider the diagram

$$(39) \quad \begin{array}{ccc} \tilde{\Pi}_m \operatorname{Rep}(G) & \xrightarrow{\tilde{\alpha}_m} & \tilde{\Pi}_m \mathcal{V}(BG) \\ & \searrow & \uparrow \Psi \cong \\ & & \pi_m \operatorname{Map}_*(BG, BU) \cong \tilde{K}^{-m}(BG). \end{array}$$

where Ψ is the isomorphism defined in Section 2.2. Our goal is to describe the diagonal map $\Psi^{-1} \circ \tilde{\alpha}_m$.

By Proposition 2.10, each class in $\tilde{\Pi}_m \operatorname{Rep}(G) \cong K_m^{\operatorname{def}}(G)$ has a representative of the form $[\rho]$ for some family of representations $\rho: S^m \rightarrow \operatorname{Rep}(G)$. Let E_ρ be the right principal $U(n)$ –bundle over $S^m \times BG$ defined by

$$\begin{aligned} E_\rho &= (S^m \times EG \times U(n)) / G \longrightarrow S^m \times BG \\ [z, e, A] &\longmapsto (z, q(e)), \end{aligned}$$

where $q: EG \rightarrow BG$ is the bundle projection and $g \in G$ acts via

$$g \cdot (z, x, A) := (z, x \cdot g^{-1}, (\rho(z)(g))A).$$

Basic properties of this construction are reviewed in Baird–Ramras [7, Section 3].

We will use $1 \in S^0 \subset S^m$ as the basepoint of S^m , and for any family

$$\rho: S^m \rightarrow \operatorname{Hom}(G, U(n)),$$

we let $\widetilde{\rho(1)}: S^m \rightarrow \operatorname{Hom}(G, U(n))$ denote the constant family with value $\rho(1): G \rightarrow U(n)$.

For based CW complexes X_1 and X_2 , the long exact sequence in K –theory for the pair $(X_1 \times X_2, X_1 \vee X_2)$ yields a (naturally) split short exact sequence

$$0 \longrightarrow \tilde{K}^0(X_1 \wedge X_2) \xrightarrow{\pi^*} \tilde{K}^0(X_1 \times X_2) \xrightarrow{i^*} \tilde{K}^0(X_1 \vee X_2) \longrightarrow 0.$$

If ρ is an S^m –family ($m > 0$), the bundle $E_\rho \rightarrow S^m \times BG$ is trivial when restricted to $S^m \times \{x\}$ (for each point $\{x\} \in BG$): indeed, each point $\tilde{x} \in q^{-1}(x) \subset EG$ gives rise to a continuous section $z \mapsto [z, \tilde{x}, I]$ of the restricted bundle. Thus we have an isomorphism

$$E_\rho|_{S^m \vee BG} \cong E_{\widetilde{\rho(1)}}|_{S^m \vee BG},$$

and hence

$$[E_\rho] - [E_{\widetilde{\rho(1)}}] \in \ker(i^*) = \operatorname{Im}(\pi^*).$$

Since π^* is injective, it follows that the class $[E_\rho] - [E_{\rho(1)}]$ has a well-defined pre-image under π^* , which we will denote by

$$(40) \quad (\pi^*)^{-1}([E_\rho] - [E_{\rho(1)}]) \in \tilde{K}^0(S^m \wedge BG) = \tilde{K}^{-m}(BG).$$

By [7, Lemma 3.1(2)], the bundles E_ρ and $E_{\rho(1)}$ only depend (up to isomorphism) on the unbased homotopy class of ρ . Hence the class (40) depends only the unbased homotopy class of ρ . Note that if ρ is constant, then $E_\rho = E_{\rho(1)}$, so in this case the class (40) is trivial.

With this understood, we have the following explicit description of $\tilde{\alpha}_*$ (or more precisely, of the map $\Psi^{-1} \circ \tilde{\alpha}_*$ in Diagram (39)).

Theorem 6.3. *Let G be a group whose classifying space BG is homotopy equivalent to a finite CW complex. Then for $m \geq 1$, the reduced topological Atiyah–Segal map, viewed as a map*

$$\tilde{\Pi}_m \text{Rep}(G) \longrightarrow \tilde{K}^{-m}(BG)$$

via the diagram (39), has the form

$$(41) \quad [\rho] \mapsto (\pi^*)^{-1}([E_\rho] - [E_{\rho(1)}]).$$

When $m = 0$, the map

$$\alpha_0: K_0^{\text{def}}(G) \cong \text{Gr}(\pi_0 \text{Rep}(G)) \longrightarrow K^0(BG)$$

is given by $\alpha_0([\rho]) = [E_\rho]$.

Note that the statement for $m = 0$ shows that the classical Atiyah–Segal map factors through α_0 , as claimed earlier.

Proof. We assume $m > 0$; the proof for $m = 0$ is similar but simpler. By definition, $\tilde{\alpha}_*([\rho]) = [B \circ \rho]$, where B is the map (37). Let $f: S^m \wedge BG \rightarrow BU(n)$ be a map classifying $(\pi^*)^{-1}([E_\rho] - [E_{\rho(1)}])$. To prove the proposition, we need to show that the adjoint map $f^\vee: S^m \rightarrow \text{Map}_*(BG, BU(n))$ satisfies

$$\Psi(\langle f^\vee \rangle) = [B \circ \rho]$$

in $\tilde{\Pi}_m \mathcal{V}(BG)$. (Recall that $\Psi(\langle f^\vee \rangle)$ is simply $[f^\vee]$.)

By choice of f , the composite $f \circ \pi$ classifies $[E_\rho] - [E_{\rho(1)}]$, and by Baird–Ramras [7, Lemma 4.1], if c is the constant map $S^m \rightarrow \text{Map}_*(BG, BU(p))$ with image $B(\rho(1))$, then $c^\vee \circ \pi$ classifies $[E_{\rho(1)}]$ (here c^\vee is the adjoint of c), while $(B \circ \rho)^\vee \circ \pi$ classifies $[E_\rho]$. Hence the maps

$$(f \circ \pi) \oplus (c^\vee \circ \pi) = (f \oplus c^\vee) \circ \pi \quad \text{and} \quad (B \circ \rho)^\vee \circ \pi$$

represent the same class in $\tilde{K}^0(S^m \wedge BG)$. Since π^* is injective, it follows that $f \oplus c^\vee$ and $(B \circ \rho)^\vee$ are homotopic as maps $S^m \wedge BG \rightarrow BU(N)$ (for sufficiently large N), and consequently $f^\vee \oplus c$ and $B \circ \rho$ are homotopic as maps $S^m \rightarrow \text{Map}_*(BG, BU(N))$. Since c is constant, it follows that $[B \circ \rho] = [f^\vee] = \Psi(\langle f^\vee \rangle)$ in $\tilde{\Pi}_m \mathcal{V}(BG)$. \square

One may replace $U(n)$ by $GL(n)$ throughout the preceding discussion, yielding a general linear version α^{GL} of the topological Atiyah–Segal map. We note that the unitary topological Atiyah–Segal map factors through this general linear version, which leads to the following natural question.

Question 6.4. Does there exist a group G for which the image of α_*^{GL} is strictly larger than the image of α_* ?

The functor $\mathcal{K}(X)$ from Section 5 is well-defined on the entire category **CGTop**, and hence α and $\tilde{\alpha}$ make sense for arbitrary discrete groups G . Our analysis of the functor \mathcal{K} , however, relied on the fact that when X is paracompact and has the homotopy type of a compact Hausdorff space, every vector bundle $E \rightarrow X$ is a direct summand of a trivial bundle.

Question 6.5. Does there exist a natural transformation of graded groups $\tilde{\eta}: \tilde{\mathcal{K}}_*(X) \rightarrow \tilde{K}^{-*}(X)$ defined for all path connected CW complexes X such that the composite

$$\tilde{K}_*(G) \xrightarrow{\alpha_*} \tilde{\mathcal{K}}_*(BG) \xrightarrow{\tilde{\eta}} \tilde{K}^{-*}(BG)$$

is described by the formula (41)?

7. RELATIONS WITH PREVIOUS WORK

In this section, we reinterpret some of the main results from Baird–Ramras [7], Ramras–Willet–Yu [39], and Ramras [43] in terms of the topological Atiyah–Segal map.

7.1. Bounds on the image of α_* . We now show that $\tilde{\alpha}_*$ fails to be surjective in dimensions $\mathbb{Q}cd(G) - 2k$ ($k > 0$), where $\mathbb{Q}cd(G)$ is the largest number n for which $H^n(G; \mathbb{Q})$ is non-zero (Theorem 7.1). This low-dimensional failure is closely analogous to the low-dimensional failure of the Quillen–Lichtenbaum conjectures (in the form discussed in [44], for instance), which relate algebraic K -theory of a field k to its étale K -theory in dimensions greater than the virtual cohomological dimension of k minus 2.

Theorem 7.1. *The image of*

$$\alpha_*: K_m^{\text{def}}(G) \rightarrow K^{-m}(BG)$$

(and, in fact, of α_^{GL}) has rank at most $\beta_m(G) + \beta_{m-2}(G) + \dots$, where $\beta_i(G)$ is the rank of $H_i(BG; \mathbb{Z})$. Hence if $\beta_d(G) \neq 0$, then the maps $\alpha_{d-2}, \alpha_{d-4}, \dots$ are not surjective.*

Proof. By Theorem 6.3 and [7, Theorem 3.5], the image of α_m lies in the subgroup of $K^{-m}(BG)$ on which the Chern classes c_{m+i} , $i = 1, 2, \dots$, vanish rationally. It follows from [7, Lemma 4.2] that the rank of this subgroup is given by the above sum of Betti numbers. The last statement follows from the fact that the Chern character is a rational isomorphism, which implies that the rank of $K^{-m}(BG)$ is equal to the sum of all the Betti numbers of BG in dimensions equivalent to m modulo 2. \square

7.2. Relation with the Novikov conjecture. Recall that a group G satisfies the strong Novikov conjecture if the analytical assembly map (from the K -homology of BG to the K -theory of the maximal C^* -algebra of G) is injective after tensoring with \mathbb{Q} . For background on this conjecture, see [39] and the references therein.

Theorem 7.2. *If G is a group such that BG is homotopy equivalent to a finite CW complex, and there exists $M > 0$ such that the unitary topological Atiyah–Segal map α_m is rationally surjective for all $m > M$, then G satisfies the strong Novikov conjecture.*

We note that Section 9 provides examples of groups G such that BG is homotopy equivalent to a finite CW complex, but α *does not* satisfy this surjectivity condition, while Theorem 7.4 provides examples in which surjectivity does hold.

Proof. Surjectivity of α_m is equivalent to surjectivity of $\tilde{\alpha}_m$, and surjectivity of $\tilde{\alpha}_m$ implies that for sufficiently large n , every element in

$$[S^m, \text{Map}_*(B\Gamma, BU(n))]$$

has the form $[B \circ \rho]$ for some S^m -family of representations ρ . It now follows from [39, Theorem 3.16] (or rather from the proof of that result) that G lies in the class of *flatly detectable groups*; all such groups satisfy the strong Novikov conjecture by [39, Corollary 4.3]. \square

7.3. Surface groups. We now translate the results in [43] into information about the topological Atiyah–Segal map when $G = \pi_1(\Sigma)$ is the fundamental group of a compact, aspherical surface Σ . Note that we allow Σ to have boundary, in which case $\pi_1(\Sigma)$ is a finitely generated free group.

We begin by recalling a result from [43], which was proven using Morse theory for the Yang–Mills functional.

Theorem 7.3 ([43], Theorem 3.4). *If Σ is a compact aspherical surface, possibly with boundary, then for each $M \geq 0$, there exists N such that for every $n > N$ the natural map*

$$B_*: \pi_m \text{Hom}(\pi_1 \Sigma, U(n)) \longrightarrow \pi_m \text{Map}_*(B\pi_1 \Sigma, BU(n))$$

induces an isomorphism on homotopy groups in dimensions $1 \leq m \leq M$ (for all choices of compatible basepoints). If Σ is non-orientable, or has non-empty boundary, then this statement holds for $0 \leq m \leq M$.

Theorem 7.4. *If Σ is a compact aspherical surface, possibly with boundary, then the topological Atiyah–Segal map*

$$\alpha_m: K_m^{\text{def}}(\pi_1 \Sigma) \longrightarrow K^{-m}(\Sigma)$$

is an isomorphism for $m \geq 1$. If Σ is non-orientable, or has non-empty boundary, then α_0 is an isomorphism as well.

Proof. Note that it suffices to prove that $\tilde{\alpha}_m$ is an isomorphism. We work in the case $m > 0$; the same reasoning will apply when $m = 0$ and Σ is non-orientable or has non-empty boundary, using the last part of Theorem 7.3.

We prove injectivity; the proof of surjectivity is similar but simpler. By Proposition 2.10, each element in the kernel of $\tilde{\alpha}_m$ has the form $[\rho]$ for some $\rho: S^m \rightarrow \text{Hom}(\pi_1 \Sigma, \text{U}(n))$ satisfying

$$(42) \quad (B \circ \rho) \oplus d \simeq c$$

for some constant maps $c, d: S^m \rightarrow \mathcal{V}(B\pi_1 \Sigma)$. Since $\mathcal{V}(B\pi_1 \Sigma)$ is stably group-like with respect to the homotopy class of the constant map

$$S^m \rightarrow \text{Map}_*(B\pi_1 \Sigma, BU(1))$$

with image $B1$ (where 1 is the trivial 1-dimensional representation of $\pi_1 \Sigma$), we may assume without loss of generality that $d \simeq BI_p$ for some p , where I_p denotes the constant map $S^m \rightarrow \text{Hom}(\pi_1 \Sigma, \text{U}(p))$ with image the trivial representation. Equation (42) now implies that $B \circ (\rho \oplus I_p)$ is nullhomotopic, and the injectivity portion of Theorem 7.3 implies that $\rho \oplus I_p$ must be nullhomotopic. Hence $[\rho] = 0$ in $\tilde{\Pi}_m(\text{Rep}(\pi_1 \Sigma))$, as desired. \square

8. FAMILIES OF FLAT CONNECTIONS OVER THE HEISENBERG MANIFOLD

In this section we study flat, unitary connections on complex vector bundles over the 3-dimensional Heisenberg manifold by combining the main results of this paper with computations due to Tyler Lawson. We begin with a review of the definition and the basic properties of this manifold.

8.1. Background. By definition, the discrete 3-dimensional Heisenberg group H is the group of 3×3 upper triangular integer matrices under ordinary matrix multiplication. This group sits as a (discrete) subgroup of the real Heisenberg group $H_{\mathbb{R}}$, which consists of all real upper triangular matrices. We will identify the real Heisenberg group with \mathbb{R}^3 via the function

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto (x, y, z)$$

(note, though, that we are using matrix multiplication to define the operation in $H_{\mathbb{R}}$, not addition in the vector space \mathbb{R}^3). The Heisenberg manifold is defined by

$$N^3 = \mathbb{R}^3 / H,$$

where H acts on $\mathbb{R}^3 \cong H_{\mathbb{R}}$ by (left) multiplication. This manifold is a Nil manifold in the sense of Thurston. We will not need this perspective here, so we refer to [45] for details.

It is an elementary exercise to check that N^3 is Hausdorff, and that the quotient map $\mathbb{R}^3 \rightarrow N^3$ is a covering map. In particular, this means N^3 is an aspherical 3-dimensional manifold with fundamental group H (and hence

$N^3 \simeq BH$), and N^3 is orientable since the action of H on \mathbb{R}^3 is orientation-preserving. Moreover, N^3 is compact; this follows, for instance, from the fact that each closed unit cube in \mathbb{R}^3 surjects onto N^3 .

Moreover, N^3 is a circle bundle over the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. Indeed, consider the mapping

$$N^3 = \mathbb{R}^3/H \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$$

given by sending $[(x, y, z)]$ to $[(x, y)]$. It is elementary to check that this map is a fiber bundle with circle fibers; indeed, for each $[(x, y)] \in \mathbb{R}^2/\mathbb{Z}^2$, there exists $\epsilon > 0$ such that the mapping

$$\begin{aligned} [x - \epsilon, x + \epsilon] \times [y - \epsilon, y + \epsilon] \times \mathbb{R}/\mathbb{Z} &\longrightarrow N^3 \\ (x', y', [z]) &\longmapsto [(x', y', z)] \end{aligned}$$

is a homeomorphism onto

$$q^{-1}(\pi([x - \epsilon, x + \epsilon] \times [y - \epsilon, y + \epsilon])),$$

where $\pi: \mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ is the quotient map.

The fibration sequence $S^1 \rightarrow N^3 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ gives rise to a short-exact sequence on fundamental groups:

$$1 \longrightarrow \mathbb{Z} \longrightarrow H \xrightarrow{q_*} \mathbb{Z}^2 \longrightarrow 1.$$

Covering space theory gives canonical identifications of $\pi_1(\mathbb{R}^3/H, [(0, 0, 0)])$ and $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, [(0, 0)])$ with H and \mathbb{Z}^2 (respectively), and under these identifications the map q_* is simply

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mapsto (a, b).$$

The kernel of q_* is generated by

$$Z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the commutator of the elements

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is elementary to check that X and Y generate H , and it follows that $\ker(q_*)$ is precisely the commutator subgroup of H , giving

$$(43) \quad H_1(N^3; \mathbb{Z}) \cong \mathbb{Z}^2.$$

Since Z commutes with both X and Y , we see that $\ker(q_*)$ is central, and it follows that H is a nilpotent group.

Poincaré Duality, together with (43), shows that the (co)homology groups of N^3 are:

$$H_i(N^3; \mathbb{Z}) \cong H^i(N^3; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 & \text{if } i = 1, 2, \\ \mathbb{Z}, & \text{if } i = 0, 3. \end{cases}$$

In particular, the cohomology of N^3 is torsion-free.

8.2. Flat bundles over the Heisenberg manifold. To understand flat bundles over N^3 , we will use the following fact.

Proposition 8.1. *If X is a finite CW complex with torsion-free integral cohomology, and $E \rightarrow X$ is a complex vector bundle whose Chern classes $c_i(E) \in H^{2i}(X; \mathbb{Z})$ vanish for $i \geq 1$, then E is stably trivial.*

In particular, if M is a smooth manifold with torsion-free integral cohomology and $E \rightarrow M$ is a vector bundle admitting a flat connection, then E is stably trivial.

Note that for finite CW complexes, $H^*(X; \mathbb{Z})$ is torsion-free if and only if $H_*(X; \mathbb{Z})$ is torsion-free.

Proof. The second statement follows from the first, because by Chern–Weil theory the Chern classes of a flat vector bundle over M map to zero in $H^*(M; \mathbb{Q})$, and when $H^*(M; \mathbb{Z})$ is torsion-free the natural map

$$H^*(M; \mathbb{Z}) \longrightarrow H^*(M; \mathbb{Q})$$

is injective (this follows by comparing the universal coefficient sequences for \mathbb{Z} and \mathbb{Q}).

To prove the first statement, consider a vector bundle $E \rightarrow X$ with $c_i(E) = 0$ for $i \geq 1$. By [4, Section 2.5] (see also [16, Proposition 6.10]) the complex X -theory of X is torsion-free (and finitely generated), so the natural map $\tilde{K}^*(X) \rightarrow \tilde{K}^*(X) \otimes \mathbb{Q}$ is injective. Composing with the Chern character gives an injection

$$\tilde{K}^*(X) \hookrightarrow \tilde{K}^*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \tilde{H}^*(X; \mathbb{Q}).$$

Since $[E]$ maps to zero under this injection, we have $[E] = 0$ in $\tilde{K}^*(X)$, so E is stably trivial as claimed. \square

Corollary 8.2. *Let G be a discrete group whose classifying space BG has torsion-free integral cohomology and has the homotopy type of a CW complex X of dimension at most d . Then if $n \geq d/2$, the bundle E_ρ associated to a representation $\rho: G \rightarrow \mathrm{GL}(n)$ is always trivial.*

Proof. By Proposition 8.1, we know that (the vector bundle associated to) E_ρ is stably trivial, meaning that its classifying map

$$B\rho: BG \simeq X \rightarrow B\mathrm{GL}(n)$$

becomes nullhomotopic after composing with the natural map

$$j: B\mathrm{GL}(n) \rightarrow B\mathrm{GL}(n + n', \mathbb{C})$$

for sufficiently large n' . Since j induces an isomorphism on homotopy groups up to dimension $2n$ and a surjection in dimension $2n + 1$, the Whitehead Theorem [30, Section 10.3] shows that for all CW complexes X of dimension at most $2n$, the map

$$[X, BGL(n)] \longrightarrow [X, BGL(n)]$$

is bijective. In particular, since $j \circ B\rho$ is nullhomotopic, we conclude that $B\rho$ is itself nullhomotopic so long as $d \leq 2n$. \square

Corollary 8.3. *The vector bundle associated to a complex representation of H is always trivial. Equivalently, every flat $GL(n)$ –bundle (and every flat $U(n)$ –bundle) over N^3 is trivial.*

Proof. Since $BH \simeq N^3$, Corollary 8.2 shows that E_ρ is trivial whenever the degree of ρ is at least 2. Since the abelianization of H is \mathbb{Z}^2 , the space of representations $\rho: H \rightarrow GL(1, \mathbb{C}) = \mathbb{C}^*$ is homeomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, and in particular is path connected. This implies that the vector bundle associated to any such ρ is isomorphic to the vector bundle associated to the trivial representation, which is a trivial vector bundle. \square

8.3. Homotopy in the space of flat connections over N^3 . Let $\mathcal{A}_n^b(N^3)$ denote the space of flat connections on the trivial bundle $N^3 \times U(n)$ (or, equivalently, the space of flat unitary connections on $N^3 \times \mathbb{C}^n$). More precisely, $\mathcal{A}_n^b(N^3)$ will denote the subspace of flat connections inside the Sobolev completion (with respect to a sufficiently strong Sobolev norm) of the space of all smooth connections on $N^3 \times U(n)$, as in [7, Section 5]. In this section we study the homotopy groups of $\mathcal{A}_n^b(N^3)$ as $n \rightarrow \infty$, using Lawson’s calculation of the deformation K –theory of $H = \pi_1(N^3)$ [22, 23, 24].

It was proven in [7, Corollary 1.3] that if M^d is a closed, smooth, aspherical d –manifold with $H^3(M; \mathbb{Q}) \neq 0$, then $\mathcal{A}_n^b(M)$ has infinitely many path components (so long as $n \geq (d+1)/2$). In particular, $\mathcal{A}_n^b(N^3)$ has infinitely many path components so long as $n \geq 2$. For manifolds M of dimension $d > 3$, [7, Corollary 1.3] also gives cohomological lower bounds on the rank of $\pi_m \mathcal{A}_n^b(M)$ for $0 < m \leq d - 3$, but for 3–manifolds no information about the homotopy groups $\pi_m \mathcal{A}_n^b(M)$ ($m \geq 1$) is obtained through the methods of that paper.

In this section, we will show that the homotopy groups of $\mathcal{A}_n^b(N^3)$ are in fact very large. Moreover, while the classes in $\pi_m \mathcal{A}_n^b(M)$ produced by the methods in [7] all admit representatives lying inside a single gauge orbit, we produce classes in $\mathcal{A}_n^b(N^3)$ that do not admit such representatives (although see Question 8.8).

This result shows a sharp contrast between the topology of spaces of flat connections over 3–manifolds and the corresponding spaces over surfaces. Let M^g denote a Riemann surface of genus g . Work of Atiyah–Bott [3], Uhlenbeck [49], Daskalopoulos [9], and Råde [38] shows that the Yang–Mills functional behaves roughly like a Morse function on the space

of connections on $E = M^g \times \mathrm{U}(n)$, and its Morse indices can be calculated using methods from complex geometry. These ideas lead to the conclusion that $\pi_* \mathcal{A}^b(E)$ vanishes for $* \leq 2g(n-1)$. Work of Ho–Liu [18, 19] extends these methods to non-orientable surfaces, leading to similar conclusions. For details and precise results, see Ramras [41, 42].

Definition 8.4. Given a space X together with a choice of representatives $\{x_C\}_{C \in \pi_0(X)}$ for the path components of X , we define

$$\tilde{\pi}_n(X) = \bigoplus_{C \in \pi_0(X)} \pi_n(X, x_C).$$

Note that up to isomorphism, this group is independent of the chosen representatives x_C .

Recall that the gauge group $\mathcal{G} = \mathrm{Map}(N^3, \mathrm{U}(n))$ acts on the space of all connections on $N^3 \times \mathrm{U}(n)$, and this action preserves the subspace $\mathcal{A}_n^b(N^3)$ (more precisely, \mathcal{G} is the Sobolev completion of the space of smooth maps with respect to the appropriate Sobolev norm). The based gauge group $\mathcal{G}_0 \leq \mathcal{G}$ is the kernel of the restriction map $\mathcal{G} \rightarrow \mathrm{U}(n)$ induced by evaluation at a fixed basepoint $x \in N^3$. The holonomy map induces a fibration sequence (in fact, a principal \mathcal{G}_0 -bundle)

$$(44) \quad \mathcal{G}_0 \xrightarrow{i_A} \mathcal{A}_n^b(N^3) \xrightarrow{\mathcal{H}ol} \mathrm{Hom}(H, \mathrm{U}(n))$$

for each n . The first map in this sequence is simply the inclusion of the gauge orbit of some flat connection $A \in \mathcal{A}_n^b(N^3)$, and sends $g \in \mathcal{G}_0$ to $g \cdot A$. We will refer to maps of the form i_A (and their induced maps on $\tilde{\pi}_*$) as *(based) gauge orbit inclusions*.

Our results about connections are an application of the following result of Tyler Lawson [24, Section 4.2].

Proposition 8.5 (Lawson). *For each $m \geq 0$, the group $K_m^{\mathrm{def}}(H)$ is free abelian of countably infinite rank.*

The proof of this result relies on spectral sequences that compute $K_*^{\mathrm{def}}(G)$ from the integral homology of spaces of irreducible unitary representations of G . The construction of the spectral sequences, reviewed in Section 9.2, makes essential use of the fact that $K_*^{\mathrm{def}}(G)$ is the homotopy of an E_∞ ring spectrum, with the ring structure arising from tensor product of representations. The representation-theoretic input for the computation comes from calculations of Lubotzky–Magid [27] and Nunley–Magid [37].

Theorem 8.6. *Given $m, R \geq 1$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $\tilde{\pi}_m(\mathcal{A}_n^b(N^3))$ contains a subgroup F satisfying:*

- (1) *The abelianization of F is free of infinite rank;*
- (2) *$\mathcal{H}ol_*(F) \leq \tilde{\pi}_m \mathrm{Hom}(H, \mathrm{U}(n))$ has rank R ; and*
- (3) *No non-trivial element in F is in the image of a based gauge-orbit inclusion map.*

When $m = 1$, and when m is even, (3) can be strengthened by replacing the based gauge group by the full gauge group.

Proof. Since $\tilde{K}^{-m}(BH) \cong K^{-m}(N^3)$ is finitely generated, Proposition 8.5 implies that the kernel of $\tilde{\alpha}_m$ is free abelian of countably infinite rank (note here that subgroups of free abelian groups – of any rank – are free [21, Appendix 2]).

Now assume $m \geq 1$. By Proposition 2.10, the natural map

$$[S^m, \text{Rep}(H)] \longrightarrow \tilde{\Pi}_m(\text{Rep}(H)) \cong \tilde{K}_m^{\text{def}}(H)$$

is surjective. This means that we can choose families

$$\rho_i: S^m \rightarrow \text{Hom}(H, \text{U}(n_i)),$$

$i = 1, 2, \dots$, such that the associated classes in $\tilde{\Pi}_m(\text{Rep}(H))$ are linearly independent and $\tilde{\alpha}_*[\rho_i] = [B \circ \rho_i]$ is zero in $\tilde{\Pi}_m(\mathcal{V}(BH))$ for each i . Since $\mathcal{V}(BH)$ is stably group-like with respect to the homotopy class of the constant map $BH \rightarrow BU(1)$, this means that

$$B \circ \rho_i: S^m \rightarrow \text{Map}_*(BH, BU(n_i))$$

becomes nullhomotopic map after composing with the natural map

$$BU(n_i) \rightarrow BU(n_i + n'_i)$$

(for sufficiently large n'_i). However, since $S^m \times BH$ is homotopy equivalent to $S^m \times N^3$, a CW complex of dimension $m + 3$, it suffices to take $n'_i \geq (m + 3)/2 - n_i$ (this is similar to the proof of Corollary 8.2). In particular, we can choose k independent of i such that

$$S^m \xrightarrow{B \circ \rho_i} \text{Map}_*(BH, BU(n_i)) \longrightarrow \text{Map}_*(BH, BU(n_i + k))$$

is nullhomotopic for each i .

Now fix an integer $R > 0$. Choose n_0 such that $n_0 \geq n_i + k$ for at least R values of the index i . Reordering the ρ_i if necessary, we can assume that $n_1 + k, \dots, n_R + k \leq n_0$. Then for $i = 1, \dots, R$, the map

$$S^m \xrightarrow{B \circ \rho_i} \text{Map}_*(BH, BU(n_i)) \longrightarrow \text{Map}_*(BH, BU(n_0))$$

is nullhomotopic, and it follows that

$$(45) \quad B \circ \rho_i \simeq B \circ \widetilde{\rho_i(1)}$$

as maps into $BU(n_0)$. The same is true for each $n \geq n_0$.

Say $n \geq n_0$, and let G be the subgroup of $\tilde{\pi}_m \text{Hom}(H, \text{U}(n))$ generated by the elements $\{[\rho_i] - \widetilde{[\rho_i(1)]}\}_{i=1}^R$ (we use additive notation, although when $m = 1$ the group $\tilde{\pi}_m \text{Hom}(H, \text{U}(n))$ may be non-abelian). Then G surjects onto the subgroup of $\tilde{\Pi}_m(\text{Rep}(H))$ generated by $\{[\rho_i]\}_{i=1}^R$, which is free abelian of rank R , so the abelianization of G must have rank R (of course when $m > 1$, the group G is already abelian).

By [7, Remark 5.6], the boundary map on homotopy groups associated to the principal bundle (44) can be identified (up to isomorphism) with the map

$$B_*: \pi_* \text{Hom}(H, \text{U}(n)) \longrightarrow \pi_* \text{Map}_*(BH, \text{BU}(n)).$$

Note here that \mathcal{G}_0 is weakly equivalent to the continuous mapping space $\text{Map}_*(N^3, \text{U}(n)) \simeq \text{Map}_*(BH, \text{U}(n))$, and $\text{Map}_*(BH, \text{BU}(n))$ is the classifying space of $\text{Map}_*(N^3, \text{U}(n))$ (this result is originally due to Gottlieb [14]). Hence

$$\pi_* \text{Map}_*(BH, \text{BU}(n)) \cong \pi_{*-1} \mathcal{G}_0$$

for $* \geq 1$. It follows from (45) that for each i , $[\rho_i] - [\widetilde{\rho_i(1)}]$ maps to zero under the boundary map for the fibration sequence (44) (with appropriately chosen basepoints), and hence there exist maps $\alpha_i: S^m \rightarrow \mathcal{A}_n^b(N^3)$ such that $\mathcal{H}ol_*([\alpha_i]) = [\rho_i] - [\widetilde{\rho_i(1)}]$.

Let A_i denote a flat connection in the image of α_i , and let $\psi_i = \mathcal{H}ol(A_i)$. Consider the long exact sequence

$$\cdots \longrightarrow \pi_1(\text{Hom}(H, \text{U}(n)), \psi_i) \xrightarrow{\partial} \pi_0 \mathcal{G}_0 \xrightarrow{(i_{A_i})^*} \pi_0 \mathcal{A}_n^b(N^3) \longrightarrow \cdots$$

Again, ∂ can be identified (up to isomorphism) with

$$B_*: \pi_1(\text{Hom}(H, \text{U}(n)), \psi_i) \longrightarrow \pi_1(\text{Map}_*(BH, \text{BU}(n)), B\psi_i),$$

and the cokernel of this map has rank 1 by [7, Corollary 1.3]. This means there exists an element $g_i \in \mathcal{G}_0$ such that the path component $[g_i]$ has infinite order in

$$\pi_0 \mathcal{G}_0 \cong \pi_1(\text{Map}_*(BH, \text{BU}(n)), B\psi_i)$$

and the subgroup of $\pi_0(\mathcal{G}_0)$ generated by $[g_i]$ intersects the image of ∂ trivially. In general, given a principal bundle K -bundle

$$K \longrightarrow P \xrightarrow{q} B,$$

if $k_1, k_2 \in K$ and $p \in P$ are elements such that $[k_1 \cdot p] = [k_2 \cdot p]$ in $\pi_0 P$, then $[k_1^{-1} k_2]$ is in the image of the boundary map

$$\pi_1(B, q(p)) \xrightarrow{\partial} \pi_0(G).$$

This means that for each fixed $i \in \{1, \dots, R\}$, the elements $[\alpha_i], [g_i \cdot \alpha_i], [g_i^2 \cdot \alpha_i], \dots$ are distinct in $\mathcal{A}_n^b(N^3)$.

Let $F \leq \widetilde{\pi}_m \mathcal{A}_n^b(N^3)$ be the subgroup generated by the elements $[g_i^k \cdot \alpha_i]$, $i = 1, \dots, R$, $k = 0, 1, \dots$. We claim that F satisfies the conditions in the theorem. First, note that $\mathcal{H}ol_*(F) = G$, so F satisfies (2).

Next, we show that the abelianization of F is freely generated by the elements $[g_i^k \cdot \alpha_i]$. For simplicity, we give the argument when $m > 1$, in which case F is already abelian; the argument for $m = 1$ just requires notational changes. If $\sum_{i,k} \lambda_{i,k} [g_i^k \cdot \alpha_i] = 0$, then summing the terms whose images lie in a particular path component C of $\mathcal{A}_n^b(N^3)$ will also give zero. By choice of the elements g_i , such a sum contains at most one term of the form $\lambda_{i,k} [g_i^k \cdot \alpha_i]$ for each i . Thus we have $\sum_i \lambda_{i,k_i} [g_i^{k_i} \cdot \alpha_i] = 0$ for some collection

of natural numbers k_i ($i = 1, \dots, R$), and every term from the original sum whose image lies in C appears in this new sum. But now

$$0 = \mathcal{H}ol_*(0) = \mathcal{H}ol_* \left(\sum_i \lambda_{i,k_i} [g_i^{k_i} \alpha_i] \right) = \sum_i \lambda_{i,k_i} [\rho_i],$$

and linear independence of the elements $[\rho_i]$ implies that $\lambda_{i,k_i} = 0$ for each i . Applying this argument to each path component, we see that all the coefficients $\lambda_{i,k}$ must be zero, as desired.

Finally, we consider gauge-orbit inclusions. Say $f \in F$ is in the image of a based gauge-orbit inclusion. We need to prove that $f = 0$. We can write $f = \sum_{i,k} \lambda_{i,k} [g_i^k \cdot \alpha_i]$ (again, for notational convenience we work in the case $m > 1$), and without loss of generality we may assume that there exists a single path component of $\mathcal{A}_n^b(N^3)$ containing $g_i^k \cdot \alpha_i(S^m)$ for all i and k . Then, as before, there can be at most one term in this sum for each i , so we can write $f = \sum_i \lambda_{i,k_i} [g_i^{k_i} \cdot \alpha_i]$. Since f is in the image of a based gauge-orbit inclusion $\mathcal{G}_0 \rightarrow \mathcal{A}_n^b(N^3)$, and the composite

$$\mathcal{G}_0 \longrightarrow \mathcal{A}_n^b(N^3) \longrightarrow \text{Hom}(H, \text{U}(n))$$

is constant, we have $\sum_i \lambda_{i,k_i} ([\rho_i] - [\widetilde{\rho_i(1)}]) = 0$. But this element maps to $\sum_i \lambda_{i,k_i} [\rho_i]$ in $\widetilde{\Pi}_m(\text{Rep}(H))$, which implies that all of the coefficients λ_{i,k_i} are in fact zero.

Finally, we consider the orbits of the full gauge group. Evaluation at a point gives a split fibration

$$(46) \quad \mathcal{G}_0 \rightarrow \mathcal{G} \rightarrow \text{U}(n),$$

and since $\text{U}(n)$ is path connected, this means that $\widetilde{\pi}_* \mathcal{G} \cong \widetilde{\pi}_* \mathcal{G}_0 \times \pi_* \text{U}(n)$ for each $*$ in \mathbb{N} .

In constructing the subgroup F , we are free to choose n_0 large with respect to m , so that $\pi_m \text{U}(n) = 0$ whenever $n \geq n_0$ and m is even. Thus when m is even, the image of a full gauge-orbit inclusion is the same (in homotopy) as the image of the corresponding based gauge-orbit inclusion.

To obtain the desired conclusion when $m = 1$, it suffices to show that for every gauge-orbit inclusion, the composite

$$\text{U}(n) \xrightarrow{s} \mathcal{G} \longrightarrow \mathcal{A}_n^b(N^3) \longrightarrow \text{Hom}(H, \text{U}(n))$$

(where s is a splitting of (46)) induces the zero map on π_1 . The image of this composite map lies inside a single conjugation orbit inside $\text{Hom}(H, \text{U}(n))$, so this composite factors through the projection $\text{U}(n) \rightarrow \text{U}(n)/Z$, where $Z \cong \text{U}(1)$ denotes the center of $\text{U}(n)$. Since the inclusion $Z \hookrightarrow \text{U}(n)$ induces isomorphisms on π_0 and π_1 , we see that $\pi_1(\text{U}(n)/Z) = 0$. \square

Remark 8.7. Despite the fact that $K_*^{\text{def}}(H)$ is extremely large, we will see in Proposition 9.9 that for odd m , the map $\alpha_*: K_m^{\text{def}}(H) \rightarrow K^{-m}(N^3)$ fails to be surjective.

Here are two natural questions regarding the above results.

Question 8.8. When m is odd and greater than 1, are the classes in $\tilde{\pi}_m \mathcal{A}_n^b(N^3)$ constructed in Theorem 8.6 in the images of gauge-orbit inclusions $\mathcal{G} \rightarrow \mathcal{A}_n^b(N^3)$?

Question 8.9. Do the results of this section extended to higher-dimensional Heisenberg manifolds?

The first step in addressing Question 8.9 would be to extend the underlying representation-theoretical work in [37].

9. MULTIPLICATIVITY OF THE TOPOLOGICAL ATIYAH–SEGAL MAP

In this final section, we explain how to enhance the topological Atiyah–Segal map into a map of E_∞ ring spectra, so that the induced map α_* on homotopy becomes a ring homomorphism. This additional structure allows us to deduce further constraints on the image of α_* for certain groups, going beyond the general bounds provided by Theorem 7.1. In particular, we obtain such results for the 3-dimensional Heisenberg group and for groups satisfying Kazhdan’s property (T).

In the case of property (T) groups, this leads to the following result regarding families of flat vector bundles.

Theorem 9.1. *Let G be a discrete group satisfying property (T), and assume that BG has the homotopy type of a finite CW complex. Consider a family of representations*

$$\rho: S^m \rightarrow \mathrm{Hom}(G, \mathrm{U}(n))$$

for some $m, n \geq 0$. Then the bundle E_ρ represents a torsion class in $\tilde{K}^0(S^m \times BG)$.

Note that by Lemma 8.1, when $H^*(BG; \mathbb{Z})$ is torsion-free the conclusion of Theorem 9.1 can be strengthened: E_ρ is in fact stably trivial.

There are many interesting groups to which this result applies. All torsion-free word hyperbolic groups admit finite CW models for BG (such a model can be built using Rips complexes - see [1, Corollary 4.12] for instance). There are many such groups with property (T), including cocompact, torsion-free lattices in $\mathrm{Sp}(n, 1)$. The fact that lattices in $\mathrm{Sp}(n, 1)$ have Property (T) is proven in [8].

9.1. Bipermutative structures. Kronecker product of matrices makes the unitary permutative action sequences giving rise to $K^{\mathrm{def}}(G)$ and $\mathcal{K}(BG)$ into *bipermutative* action sequences, in the sense described in Section 3.3; the details are just a routine extension of the computations in May [33, VI §5]. We thus obtain functors K_\otimes^{def} and \mathcal{K}_\otimes from the category of discrete groups to the category of E_∞ ring spectra, which become naturally equivalent to K^{def} and \mathcal{K} after applying the forgetful functor to spectra.

Theorem 9.2. *There is a natural transformation α^\otimes between the functors K_\otimes^{def} and \mathcal{K}_\otimes , which becomes equivalent to α after applying the forgetful functor from E_∞ ring spectra to spectra. In particular, α_* is a homomorphism of unital rings, and $\tilde{\alpha}_*$ is a homomorphism of non-unital rings.*

Proof. The desired natural transformation is again induced by the simplicial classifying space functor B , which respects the multiplicative structure as well as the additive structures (by functoriality, essentially). The statement regarding $\tilde{\alpha}_*$ follows from the fact that $\tilde{K}_*(G)$ and $\tilde{K}_*(BG)$ are simply the kernels of the compatible (ring) homomorphisms induced by the inclusion $\{1\} \rightarrow G$. So $\tilde{\alpha}_*$ is just the induced map between these ideals. \square

The defect in this construction is that while the homotopy groups $\mathcal{K}_*(X)$ agree *additively* with the complex topological K -theory of X , the ring structure is not immediately accessible in general. Specifically, I do not know whether this ring always satisfies Bott periodicity. Nevertheless, applications of Theorem 9.2 are provided in Section 9 below, based on the following (rather limited) information regarding the rings $\mathcal{K}_*(X)$. When $X = \{*\}$, May [33, VIII §2] showed that the ring $\mathcal{K}_*(*)$ is isomorphic to $\pi_*\mathbf{ku} = \mathbb{Z}[\beta]$, where $\beta \in \pi_2(\mathbf{ku}) \cong \mathbb{Z}$ is a generator. In other words, this is the standard Bott-periodic connective K -theory ring of a point. For each finite CW complex X , the injective ring map $\mathcal{K}_*(*) \rightarrow \mathcal{K}_*(X)$ (induced by the projection $X \rightarrow *$) now embeds the ring $\pi_*\mathbf{ku}$ in $\mathcal{K}_*(X)$.

Question 9.3. Is the additive isomorphism

$$\bigoplus_{m=0}^{\infty} K_m^{\text{def}}(X) \cong \bigoplus_{m=0}^{\infty} K^{-m}(X)$$

an isomorphism of rings, where the latter graded group has the ring structure induced by tensor product of vector bundles? More specifically, is there an isomorphism as above, induced by a natural map of E_∞ ring spectra from $K^{\text{def}}(X)$ to the function spectrum $F(X_+, \mathbf{ku})$? (Here X_+ denotes X with a disjoint basepoint.)

We note that there is an unbased version of the topological monoid $\mathcal{V}(X)$, namely

$$\text{Map} \left(X, \prod_{n=0}^{\infty} BU(n) \right),$$

which supports a multiplicative structure more closely related to the multiplication in K -theory. I do not know how to relate this monoid to $\mathcal{V}(X)_{hU}$ in general, however.

9.2. Deformation K -theory and spaces of irreducible representations. We need to review some of Lawson's results from [23, 24], which allow one to compute (unitary) deformation K -theory from homological information about spaces of irreducible representations.

First, consider the space

$$\overline{\text{Rep}}(G) = \coprod_n \text{Hom}(G, \text{U}(n))/\text{U}(n),$$

where the quotient on the right is taken with respect to the conjugation action. Block sum makes this into a strictly commutative topological monoid. In fact, the sequence of spaces

$$\text{Rep}_n(G) = \text{Hom}(G, \text{U}(n))/\text{U}(n)$$

form a *bipermutative* action sequence for the trivial groups $G_n = \{1\}$ (using Kronecker product to define the multiplicative structure) and the associated E_∞ ring spectrum $R^{\text{def}}(G)$ satisfies $\Omega^\infty R^{\text{def}}(G) \simeq \overline{\text{Rep}}(G)$ by Proposition 3.6. We define

$$R_*^{\text{def}}(G) = \pi_* R^{\text{def}}(G).$$

The quotient maps $\text{Hom}(G, \text{U}(n)) \rightarrow \text{Hom}(G, \text{U}(n))/\text{U}(n)$ respect block sum and Kronecker product, so we obtain an induced map of E_∞ ring spectra

$$K^{\text{def}}(G) \longrightarrow R^{\text{def}}(G).$$

At this point, we need to pass from the category of E_∞ ring spectra to the category of \mathbf{S} -algebras, as constructed in [13]. The desired functor is discussed in [13, II.3], and for us the important point is that it induces an isomorphism on the underlying homotopy rings. We will continue to use the same notation for our E_∞ ring spectra and their associated \mathbf{S} -algebras, but it should be noted that smash products will be formed in the derived category of \mathbf{S} modules or \mathbf{ku} -modules, as appropriate.

In [23], it is shown that when G is finitely generated, there is an equivalence $H\mathbb{Z} \wedge K^{\text{def}}(G) \simeq R^{\text{def}}(G)$. Fix a generator of $\pi_2 \mathbf{ku}$ and a map $\mathbf{S}^2 \rightarrow \mathbf{ku}$ representing it. Smashing this map with \mathbf{ku} induces a map

$$\beta: \Sigma^2 \mathbf{ku} \longrightarrow \mathbf{ku}$$

which we call the Bott map. Bott periodicity implies that the homotopy cofiber of β is the Eilenberg–MacLane spectrum $H\mathbb{Z}$.

Smashing the homotopy cofiber sequence

$$\Sigma^2 \mathbf{ku} \longrightarrow \mathbf{ku} \longrightarrow H\mathbb{Z}$$

with $K^{\text{def}}(G)$ (as \mathbf{ku} -modules; that is, applying $\wedge_{\mathbf{ku}}$) and taking homotopy groups now gives a long exact sequence of the form

$$(47) \dots \xrightarrow{\partial} K_*^{\text{def}}(G) \xrightarrow{\beta} K_{*+2}^{\text{def}}(G) \longrightarrow R_{*+2}^{\text{def}}(G) \xrightarrow{\partial} K_{*-3}^{\text{def}}(G) \xrightarrow{\beta} \dots$$

Lawson also developed a spectral sequence for computing $R_*^{\text{def}}(G)$ from the integral homology of the spaces

$$\text{Irr}_n^+(G) := \text{Rep}_n(G)/\text{Sum}_n(G),$$

where $\text{Sum}_n(G)$ denotes the subspace of reducible representations. Note that $\text{Irr}_n^+(G)$ is the one-point compactification of complement of $\text{Sum}_n(G)$ in $\text{Rep}_n(G)$, and this complement is precisely the subspace of irreducible

representations. The spectral sequence is constructed by considering the tower of spectra

$$* = R_{\leq 0}^{\text{def}}(G) \longrightarrow R_{\leq 1}^{\text{def}}(G) \longrightarrow R_{\leq 2}^{\text{def}}(G) \longrightarrow \cdots,$$

where $R_{\leq k}^{\text{def}}$ is the spectrum associated to the subspaces of $\text{Rep}_n(G)$ consisting of representations whose irreducible summands all have dimension at most k ; note that these subspaces provide a submonoid of $\overline{\text{Rep}(G)}$, and in fact a permutative sequence of the sequence $(\text{Rep}_n(G))_{n=0}^\infty$. The homotopy colimit of this sequence is $R^{\text{def}}(G)$, and Lawson proves that there are homotopy cofiber sequences of spectra

$$R_{k-1}^{\text{def}}(G) \longrightarrow R_k^{\text{def}}(G) \longrightarrow H\mathbb{Z} \wedge \text{Irr}_k^+(G)$$

for each $k \geq 1$.

In general, a sequence of spectra

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

gives rise to an exact couple

$$\begin{array}{ccc} \bigoplus_{q,p} \pi_q X_p & \xrightarrow{\oplus (f_p)_*} & \bigoplus_{q,p} \pi_q X_p \\ & \swarrow \partial \quad \searrow & \\ & \bigoplus_{q,p} \pi_q (\text{hocofib } f_p) & \end{array}$$

and hence to a spectral sequence of the form

$$E_{p,q}^1 = \pi_{p+q}(\text{hocofib } f_p) \implies \pi_{p+q} \text{hocolim}_i X_i,$$

with differentials

$$d_{p,q}^r : E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r.$$

Since $H\mathbb{Z}$ is the spectrum representing integral homology, in the case at hand we obtain a spectral sequence

$$(48) \quad E_{p,q}^1 = \tilde{H}_{p+q}(\text{Irr}_p^+(G); \mathbb{Z}) \implies \pi_{p+q} R^{\text{def}}(G).$$

Note that with this indexing, the spectral sequence can be non-zero in the quadrant where $p, q \geq 0$ and in the region where $-p \leq q < 0$.

9.3. Groups satisfying Kazhdan's property (T). Property (T) has been widely studied since its introduction by Kazhdan in the late 1960s. Loosely speaking, property (T) is a weak rigidity property for (possibly infinite-dimensional) unitary representations of locally compact groups. For a broad introduction to this subject, see [8]. We note that every discrete group with property (T) is finitely generated (this is a result of Kazhdan; for a proof see [8, Theorem 1.3.1]). Hence we can utilize the results of Lawson discussed above.

We need a lemma regarding finite-dimensional unitary representations of property (T) groups.⁴

Lemma 9.4. *Let G be a discrete group with property (T). Then the space $\text{Hom}(G, \text{U}(n))/\text{U}(n)$ is a finite, discrete space.*

Proof. This is a simple consequence of a theorem on S. P. Wang [50, Theorem 2.5], which states that if $\rho: G \rightarrow \text{U}(n)$ is an irreducible representation of a group with property (T), then the path component of ρ in $\text{Hom}(G, \text{U}(n))$ coincides with its conjugation orbit O_ρ .

Since $\text{Hom}(G, \text{U}(n))$ is compact and triangulable, it has finitely many path components, and hence Wang's theorem implies that there are only finitely many irreducible unitary representations in each dimension. Since every unitary representation is a direct sum of irreducibles, finiteness of $\text{Hom}(G, \text{U}(n))/\text{U}(n)$ follows immediately, and discreteness follows as well since this space is Hausdorff. \square

Lemma 9.5. *Let G be a discrete group satisfying property (T). Then $K_{2m+1}^{\text{def}}(G)$ is trivial for all $m \geq 0$, and the iterated Bott map*

$$\beta_*^m: K_0^{\text{def}}(G) \cong \text{Gr}(\pi_0 \text{Rep}(G)) \longrightarrow K_{2m}^{\text{def}}(G)$$

is an isomorphism for all $m \geq 1$.

Proof. By Lemma 9.4, the space $\text{Hom}(G, \text{U}(n))/\text{U}(n)$ is discrete for every n , and it follows that the same is true for $\text{Irr}_n^+(G)$. Hence the homology of these spaces vanishes in positive dimensions, and the spectral sequence (48) implies that $\pi_*(R^{\text{def}}(G)) = 0$ for $* > 0$. From the long exact sequence (47), we now see that $K_1^{\text{def}}(G) \cong R_1^{\text{def}}(G) = 0$, and that the Bott map $\beta_*: K_m^{\text{def}}(G) \longrightarrow K_{m+2}^{\text{def}}(G)$ is an isomorphism for all $m \geq 0$. \square

We need a standard fact regarding flat bundles, which follows from Chern–Weil theory.

Lemma 9.6. *Let G be a discrete group such that BG has the homotopy type of a finite CW complex, and let ϵ^n denote the trivial bundle $BG \times \text{U}(n)$. Then for every representation $\rho: G \rightarrow \text{U}(n)$, the class $[E_\rho] - [\epsilon^n]$ is torsion in $\tilde{K}^0(BG)$.*

A complete proof (of a much more general statement, in fact) can be found in [7, Theorem 3.5].

Proposition 9.7. *Let G be a discrete group satisfying property (T), and assume that BG has the homotopy type of a finite CW complex. Then the reduced unitary topological Atiyah–Segal map*

$$\tilde{\alpha}_m: \tilde{K}_*^{\text{def}}(G) \longrightarrow \tilde{K}^{-m}(G)$$

is zero when m is odd, and its image is torsion when m is even.

⁴I learned this result from Rufus Willett.

Proof. For m odd, this is immediate from Lemma 9.5, so we consider the even case.

Let G be a group satisfying the hypotheses. By Theorem 6.3, the image of α_0 consists of the K -theory classes of the form $[E_\rho]$, where $\rho: G \rightarrow \mathrm{U}(n)$ is a single representation. Lemma 9.6 implies that the image of α_0 becomes torsion after modding out the summand $\pi_0 \mathbf{ku} \cong \mathbb{Z}$ corresponding to the trivial bundles. But since $\alpha_0 = \tilde{\alpha}_0 \oplus \mathrm{Id}_{\pi_0 \mathbf{ku}}$ (see (38)), this quotient is isomorphic to the image of $\tilde{\alpha}_0$.

Now consider the commutative diagram

$$(49) \quad \begin{array}{ccc} K_{2m}^{\mathrm{def}}(G) & \xrightarrow{\alpha_{2m}} & \mathcal{K}_{2m}(BG) \\ \beta_*^m \uparrow \cong & & \uparrow \cdot \alpha_0(\beta^m) \\ K_0^{\mathrm{def}}(G) & \xrightarrow{\alpha_0} & \mathcal{K}_0(BG), \end{array}$$

where the map on the right is multiplication by $\alpha_0(\beta^m) \in \pi_{2m} \mathcal{K}(BG)$. Since $K_*^{\mathrm{def}}(BG)$ is a ring, this map is a group homomorphism. Each group in the diagram contains a \mathbb{Z} summand arising from the homotopy of \mathbf{ku} , via the maps induced by $G \rightarrow \{1\}$ and $BG \rightarrow \{*\}$, and these summands are complementary to the reduced subgroups. All four maps in the diagram are isomorphisms when restricted to these \mathbb{Z} summands, so the image of $\beta^m \circ \alpha_0$, and hence also of $\alpha_{2m} \circ \beta^m$, has rank 1. Since $\beta_*^m: K_0^{\mathrm{def}}(G) \rightarrow K_{2m}^{\mathrm{def}}(G)$ is an isomorphism, we conclude that the image of α_{2m} has rank 1, and finally that the image of $\tilde{\alpha}_{2m}$ is torsion, as desired. \square

We can now prove the promised result regarding families of flat bundles.

Proof of Theorem 9.1. By Proposition 9.7 and Theorem 6.3, we know that the class

$$\tilde{\alpha}_m([\rho]) = \pi_*^{-1} \left([E_\rho] - [E_{\rho(1)}] \right)$$

is torsion in $\tilde{K}^0(S^m \wedge BG)$. But

$$\pi_*: \tilde{K}^0(S^m \wedge BG) \longrightarrow \tilde{K}^0(S^m \times BG)$$

is (split) injective, so for some $k \geq 1$ we have

$$k[E_\rho] = k[E_{\rho(1)}] \text{ in } K^0(S^m \times BG).$$

Hence it will suffice to show that $l[E_{\rho(1)}]$ represents a torsion class in reduced K -theory for some $l \geq 1$. This holds for the bundle $E_{\rho(1)} \rightarrow BG$ by Lemma 9.6, and since $E_{\rho(1)}$ is a pullback of $E_{\rho(1)}$, the proof is complete. \square

It is tempting to attempt to prove Theorem 9.1 directly from Lemma 9.4, which implies that the constituent representations $\rho(z)$ in a family ρ are all isomorphic as z varies over the sphere. This can be done when $\rho(z)$ is irreducible (as we explain), but it is not clear that this condition gives useful information in general: first, in any bundle over $S^m \times X$, the restrictions to $\{z\} \times X$ are all isomorphic; and second, there is no guarantee that one can

find a continuous family of matrices A_z satisfying $A\rho(z)A^{-1} = \rho(1)$ for all $z \in S^m$. When such a family A exists, it is a simple matter to check that $E_\rho \cong E_{\rho(1)}$. However, such a family exists (up to homotopy), if and only if $[\rho]$ is in the image of the map

$$(50) \quad [S^m, \mathrm{U}(n)] \longrightarrow [S^m, O_\rho]$$

induced by the quotient map $\mathrm{U}(n) \rightarrow \mathrm{U}(n)/\mathrm{Stab}(\rho) \cong O_\rho$. But this map often fails to be surjective; for instance when m is even and $m < 2n$, we have $\pi_m(\mathrm{U}(n)) = 0$, and

$$\pi_m(O_\rho) \cong \ker(\pi_{m-1}(\mathrm{Stab}(\rho)) \rightarrow \pi_{m-1}\mathrm{U}(n)).$$

When ρ has more than one isotypical component, this kernel is non-trivial, and this prevents (50) from being surjective. On the other hand, when ρ is irreducible, we have $O_\rho \cong \mathrm{PU}(n)$, the projective unitary group, and the map $\pi_m\mathrm{U}(n) \rightarrow \pi_m\mathrm{PU}(n)$ is indeed surjective for all $m \geq 0$, which gives an elementary proof of Theorem 9.1 in this case.

Remark 9.8. Theorem 9.1 provides a partial answer to [39, Question 3.20]. If G satisfies the hypotheses of Theorem 9.1, then no non-trivial class in the *reduced* rational K -homology of BG can be detected by a spherical family of representations (in the sense of [39, Definition 3.4]). However, it remains possible that non-trivial classes can be detected by non-spherical families of representations.

9.4. The topological Atiyah–Segal map for the Heisenberg group.

In the case of the 3-dimensional integral Heisenberg group H , Lawson showed in [22, 23] that $R_m^{\mathrm{def}}(H) = 0$ for $m \geq 3$. In dimension 1, Theorem 7.1 tells us that the image of $\alpha_1 = \alpha_1^H$ has rank at most $\beta_1(N^3) = 2$. Reasoning similar to the proof of Proposition 9.7 yields the following result.

Proposition 9.9. *The image of the unitary topological Atiyah–Segal map*

$$\alpha_{2m+1}: K_{2m+1}^{\mathrm{def}}(H) \longrightarrow K^{-(2m+1)}(BH) \cong K^{-(2m+1)}(N^3)$$

has rank at most 2 for each $m \geq 0$. In particular, α_ is never surjective in odd dimensions.*

The failure of surjectivity is somewhat surprising, since as discussed in Section 8, Lawson showed that the deformation K -theory of H is free abelian of infinite rank in each degree.

We note that the image of α_0^H has rank 1 by Theorem 7.1. It would be interesting to calculate the ranks of α_1^H and α_2^H .

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